MATH2111 Higher Several Variable Calculus
Analysis

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Analysis

Concepts from real one-variable calculus

$f : \mathbb{R} \to \mathbb{R}$

- limits
- continuity
- differentiability
- integrability

Theorems

- Min/Max
  A continuous function on a closed interval attains a max and min value.
- Intermediate Value Theorem
  A continuous function on $[a, b]$ attains all values in $[f(a), f(b)]$.
- Mean Value Theorem
  Connects the instantaneous rate of change of a differentiable function to its change over a finite closed interval.

Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$

We want to study functions with domain $D \subset \mathbb{R}^n$

$$\begin{array}{lr} f : D \to \mathbb{R} & \text{scalar fields} \\ \mathbf{f} : D \to \mathbb{R}^m & \text{vector fields} \end{array}$$

Examples

$$\mathbf{f}(\mathbf{x}) = A\mathbf{x} \qquad \text{where } A \text{ is a matrix}$$

$$\mathbf{f}(x_1, x_2, x_3) = (x_1^2 - x_2^2, -\sin x_2)$$

A vector field can be thought of as m scalar fields, its components, that is

$$\mathbf{f}(\mathbf{x}) = \Big(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x})\Big).$$

$f_1, f_2, \ldots, f_m$ are the components of $\mathbf{f}$.

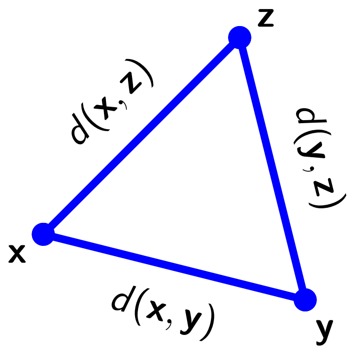
Distance Functions (metrics)

The usual Euclidean distance between $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ is

$$d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

It's easy to check that this satisfies

1. $d(\mathbf{x}, \mathbf{y}) \geq 0 \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $d(\mathbf{x}, \mathbf{y}) = 0 \Leftrightarrow \mathbf{x} = \mathbf{y}$
2. $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x}) \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$
3. $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) \quad \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$.



A function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ statisfying these properties is called a metric.

A metric is (1) positive definite, (2) symmetric and (3) satisfies the triangle inequality.

Distance Functions (metrics)

Examples

- $d_p(\mathbf{x},\mathbf{y}) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}$ for $1 \le p < \infty$.

$$\begin{aligned} \text{eg} \qquad d_1(\mathbf{x},\mathbf{y}) &= |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n| \\ d_2(\mathbf{x},\mathbf{y}) &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} \end{aligned}$$

- $d_\infty = \max\Big(|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\Big)$

Eg,

$$\begin{aligned} d_1\Big((1,2,3),(-1,2,4)\Big) &= |1-(-1)| + |2-2| + |3-4| &&= 3 \\ d_2\Big((1,2,3),(-1,2,4)\Big) &= \sqrt{(1-(-1))^2 + (2-2)^2 + (3-4)^2} &&= \sqrt{5} \\ d_\infty\Big((1,2,3),(-1,2,4)\Big) &= \max(2,0,1) &&= 2 \end{aligned}$$

Metrics

A related concept of a norm (length of an element in a vector space) will be used in the Fourier series section of the course. It's definition is not given here.

If $||\mathbf{x}||$ is the norm of $\mathbf{x} \in \mathbb{R}^n$, then for $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$, $d(\mathbf{x},\mathbf{y}) = ||\mathbf{x}-\mathbf{y}||$ defines a metric.

Definition

Two metrics d and δ are equivalent if there exists constants

$$0 < c < C < \infty$$

such that

$$c\delta(\mathbf{x},\mathbf{y}) \le d(\mathbf{x},\mathbf{y}) \le C\delta(\mathbf{x}.\mathbf{y}) \qquad \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^n.$$

[This is an equivalence relation as studied in MATH1081.]

Problem 8 on tutorial sheet 2 shows d_2 and d_∞ are equivalent. (d_p and d_∞ are also equivalent.)

Limits of sequences

Definition

A ball around $\mathbf{a} \in \mathbb{R}^n$ of radius $\epsilon > 0$ is the set

$$B(\mathbf{a}, \epsilon) = \{\mathbf{x} \in \mathbb{R}^n \ : \ d(\mathbf{a}, \mathbf{x}) < \epsilon\}.$$

Think of "$\mathbf{x}$ is close to $\mathbf{a}$" as meaning that $\mathbf{x} \in B(\mathbf{a}, \epsilon)$ for some small postive ϵ.

Definition

For a sequence $\{\mathbf{x}_i\}$ of points in $\mathbb{R}^n$ we say $\mathbf{x}$ is the limit of the sequence $\{\mathbf{x}_i\}$ if and only if

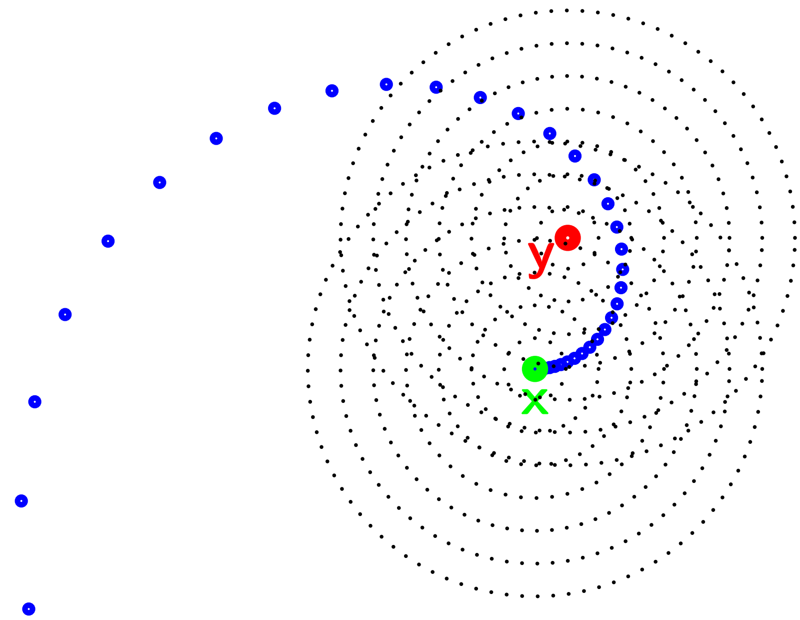
$$\forall \epsilon > 0 \ \exists N \text{ such that } n \geq N \Rightarrow d(\mathbf{x}, \mathbf{x}_n) < \epsilon$$

or equivalently

$$\forall \epsilon > 0 \ \exists N \text{ such that } n \geq N \Rightarrow \mathbf{x}_n \in B(\mathbf{x}, \epsilon).$$

If $\mathbf{x}$ is the limit of the sequence $\{\mathbf{x}_i\}$ then for each postive ϵ there is a point in the sequence beyond which all points of the sequence are inside $B(\mathbf{x}, \epsilon)$.

Limits of sequences



$\mathbf{x}$ is the limit of the sequence of blue dots.

$\mathbf{y}$ is the not limit of the sequence of blue dots.

If $\mathbf{x}$ is the limit of the sequence $\mathbf{x}_i$ then for all positive ϵ there are always infinitely many points of the sequence inside $B(\mathbf{x}, \epsilon)$.

[The converse is not true. Can you think of a counter example?]

Limits of sequences

Sketch the following.

- The unit ball in $\mathbb{R}^2$ using d_1
- The unit ball in $\mathbb{R}^2$ using d_2
- The unit ball in $\mathbb{R}^2$ using d_3
- The unit ball in $\mathbb{R}^2$ using d_4
- The unit ball in $\mathbb{R}^2$ using d_∞
- The unit ball in $\mathbb{R}^3$ using d_2
- The unit ball in $\mathbb{R}^3$ using d_∞

Limit example

Let $\mathbf{x}_k = \left(2 - \frac{1}{k}, e^{-k}\right)$, $k = 1, 2, 3, \ldots$. Show that $\lim_{k\to\infty} \mathbf{x}_k = (2, 0)$.

Let $\mathbf{x} = (2, 0)$. How big does k need to be to ensure that $\mathbf{x}_k \in B(\mathbf{x}, \epsilon)$, $\mathbf{x}_{k+1} \in B(\mathbf{x}, \epsilon), \ldots$ etc?

$$d(\mathbf{x}, \mathbf{x}_k) = \sqrt{\left(2 - \frac{1}{k} - 2\right)^2 + \left(e^{-k} - 0\right)^2} = \sqrt{\frac{1}{k^2} + e^{-2k}}$$

[We don't need the smallest k that makes $\mathbf{x}_k, \mathbf{x}_{k+1}, \mathbf{x}_{k+2} \in B(\mathbf{x}, \epsilon)$.]

Since $\frac{1}{k^2} > e^{-2k}$ for $k \geq 1$,

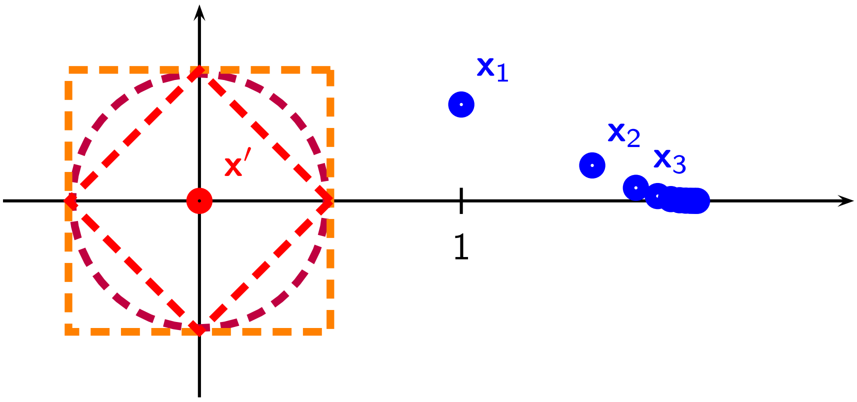
$$d(\mathbf{x}, \mathbf{x}_k) < \sqrt{\frac{1}{k^2} + \frac{1}{k^2}} = \frac{\sqrt{2}}{k}.$$

So if we take $K = \left\lceil \frac{\sqrt{2}}{\epsilon} \right\rceil$ then

$$k > K \;\Rightarrow\; d(\mathbf{x}, \mathbf{x}_k) < \frac{\sqrt{2}}{k} < \frac{\sqrt{2}}{K} \leq \epsilon \;\Rightarrow\; \mathbf{x}_k \in B(\mathbf{x}, \epsilon).$$

Limit example

Show that $\mathbf{x}' = (0,0)$ is not the limit of $\mathbf{x}_k = \left(2 - \frac{1}{k}, e^{-k}\right)$.



$\forall \epsilon > 0\ \exists K$ such that

$k \geq K \ \Rightarrow\ \mathbf{x}_k \in B(\mathbf{x}, \epsilon)$.

Using d_1: $B_1(\mathbf{x}, \frac{1}{2})$ does not contain infinitely many members of the sequence.

Using d_2: $B_2(\mathbf{x}, \frac{1}{2})$ does not contain infinitely many members of the sequence.

Using d_∞: $B_\infty(\mathbf{x}, \frac{1}{2})$ does not contain infinitely many members of the sequence.

The limit of the sequence is not $\mathbf{x}'$ for any of these equivalent metrics.

Limits of sequences

Theorem

A sequence $\mathbf{x}_k$ converges to a limit $\mathbf{x}$

$\Leftrightarrow$ *the components of $\mathbf{x}_k$ converge to the components of $\mathbf{x}$.*

$\Leftrightarrow$ $d(\mathbf{x}_k, \mathbf{x}) \to 0$.

We can use any equivalent distance function (metric). Why? See tutorial sheet 2 problems 7 and 8.

Limits and equivalent metrics

Suppose d and δ are two equivalent metrics. That is,

$$cd(\mathbf{x}, \mathbf{y}) \leq \delta(\mathbf{x}, \mathbf{y}) \leq Cd(\mathbf{x}, \mathbf{y})$$

for some strictly positive constants c and C.

Now, using d as the metric, suppose

$$\mathbf{x}_k \to \mathbf{x} \qquad \text{for } \mathbf{x}_k, \mathbf{x} \in \mathbb{R}^n$$

that is,

$$\forall \epsilon > 0 \ \exists K \text{ such that } k \geq K \Rightarrow d(\mathbf{x}_k, \mathbf{x}) < \epsilon \qquad (*)$$

We want to make a similar statement using δ.

$\forall \epsilon' > 0$ choose ϵ so that $\epsilon' = C\epsilon$. Since $\epsilon > 0$, $(*)$ says $\exists K$ such that

$$k \geq K \ \Rightarrow \ d(\mathbf{x}_k, \mathbf{x}) < \epsilon \ \Rightarrow \ \delta(\mathbf{x}_k, \mathbf{x}) \leq Cd(\mathbf{x}_k, \mathbf{x}) < C\epsilon = \epsilon'$$

that is $\delta(\mathbf{x}_k, \mathbf{x}) < \epsilon'$. Hence $\mathbf{x}_k \to \mathbf{x}$ using the metric δ.

Proof of first part of theorem ($\Rightarrow$)

We can use any equivalent metric. Let's use d_∞.

Suppose $\mathbf{x}_k \to \mathbf{x}$ for $\mathbf{x}_k, \mathbf{x} \in \mathbb{R}^n$, that is

$$\forall \epsilon > 0 \ \exists K \text{ such that } k \geq K \ \Rightarrow \ d_\infty(\mathbf{x}_k, \mathbf{x}) < \epsilon.$$

Now, for any $i = 1, 2, \ldots, n$, (ie for any component)

$$\begin{aligned} |x_{k,i} - x_i| \ &\leq \ \max\Big(|x_{k,1} - x_1|, |x_{k,2} - x_2|, \ldots, |x_{k,n} - x_k|\Big) \\ &= \ d_\infty(\mathbf{x}_k, \mathbf{x}). \end{aligned}$$

Hence $\forall \epsilon > 0$, there is a K (the same K as above) such that

$$k \geq K \ \Rightarrow \ |x_{k,i} - x_i| < \epsilon$$

and so

$$x_{k,i} \to x_i.$$

Proof of first part of theorem ($\Leftarrow$)

If all of the components of $\mathbf{x}_k$ converge, then $\forall \epsilon > 0$,

$$\begin{aligned}
&\exists K_1 \text{ such that } k \geq K_1 \;\Rightarrow\; |x_{k,1} - x_1| < \epsilon \\
&\exists K_2 \text{ such that } k \geq K_2 \;\Rightarrow\; |x_{k,2} - x_2| < \epsilon \\
&\vdots \\
&\exists K_n \text{ such that } k \geq K_n \;\Rightarrow\; |x_{k,n} - x_n| < \epsilon
\end{aligned}$$

If we take $K = \max(K_1, K_2, \ldots, K_n)$ then

$$\begin{aligned}
&k \geq K \;\Rightarrow\; |x_{k,1} - x_1| < \epsilon \\
&k \geq K \;\Rightarrow\; |x_{k,2} - x_2| < \epsilon \\
&\vdots \\
&k \geq K \;\Rightarrow\; |x_{k,n} - x_n| < \epsilon
\end{aligned}$$

$$\Rightarrow d_\infty(\mathbf{x}_k, \mathbf{x}) = \max\Big(|x_{k,1} - x_1|, |x_{k,2} - x_2|, \ldots, |x_{k,n} - x_k|\Big) < \epsilon.$$

Hence

$$\forall \epsilon > 0 \; \exists K \text{ such that } k \geq K \;\Rightarrow\; d_\infty(\mathbf{x}_k, \mathbf{x}) < \epsilon.$$

Cauchy sequences

We can define convergence without knowing the limit of a sequence.

Definition

A sequence $\{\mathbf{x}_k\}$ in $\mathbb{R}^n$ is a Cauchy sequence if

$$\forall \epsilon > 0 \; \exists K \text{ such that } k, l > K \;\Rightarrow\; d(\mathbf{x}_k, \mathbf{x}_l) < \epsilon.$$

Theorem

A sequence $\{\mathbf{x}_k\}$ converges in $\mathbb{R}^n$ $\Leftrightarrow$ $\{\mathbf{x}_k\}$ is a Cauchy sequence.

Proof.

"$\Rightarrow$" $\quad d(\mathbf{x}_k, \mathbf{x}_l) \leq d(\mathbf{x}_k, \mathbf{x}) + d(\mathbf{x}, \mathbf{x}_l) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

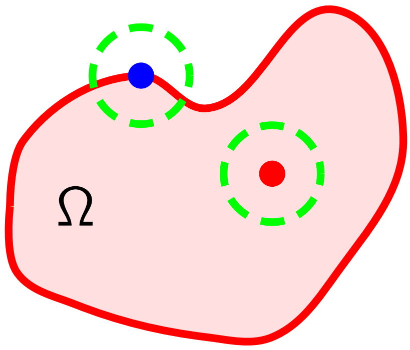
"$\Leftarrow$" This depends on how $\mathbb{R}$ is constructed. One common definition of $\mathbb{R}$ is that it is the set of limits of Cauchy sequences. $\square$

Open and closed sets

Definition

Consider $\Omega \subset \mathbb{R}^n$.

- $\mathbf{x}_0 \in \Omega$ is an interior point of Ω if there is a ball around $\mathbf{x}_0$ contained in Ω.
- Ω is open if every point of Ω is an interior point.
- Ω is closed if its complement is open.
- $\mathbf{x}_0 \in \mathbb{R}^n$ is a boundary point of Ω if every ball around $\mathbf{x}_0$ contains both points in Ω and points not in Ω.



Theorem

$\Omega \subset \mathbb{R}^n$ *is closed* $\Leftrightarrow$ *it contains all of its boundary points.*

Examples of open and closed sets

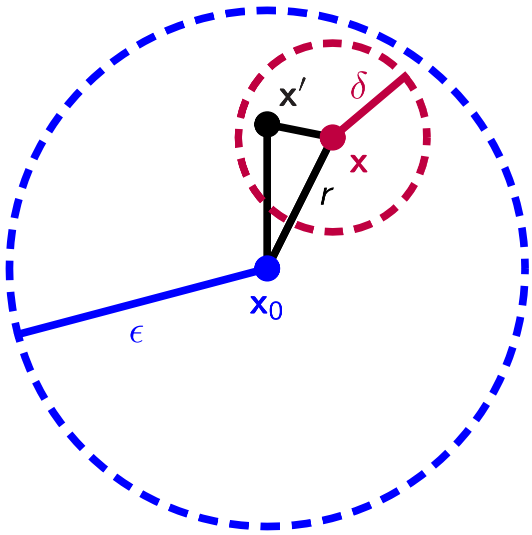
on $\mathbb{R}$	$[a, b]$	closed
	(a, b)	open
	$\mathbb{R}$	open and closed
	$\emptyset$	open and closed
	$[a, b)$	neither
	$\mathbb{Q}$	neither
	$\{k^{-1} : k \in \mathbb{Z}^+\}$	neither
on $\mathbb{R}^n$	$B(\mathbf{x}_0, \epsilon)$	open

Open and closed sets

Prove that for $\mathbf{x}_0 \in \mathbb{R}^n$ and $\epsilon \in \mathbb{R}^+$

$$B(\mathbf{x}_0, \epsilon) = \{\mathbf{x} \in \mathbb{R}^n \ : \ d(\mathbf{x}, \mathbf{x}_0) < \epsilon\}$$

is an open subset of $\mathbb{R}^n$.



For each $\mathbf{x} \in B(\mathbf{x}_0, \epsilon)$ we need to show that there is a δ such that $B(\mathbf{x}, \delta) \subset B(\mathbf{x}_0, \epsilon)$.

Choose $\delta = \epsilon - r$ where $r = d(\mathbf{x}_0, \mathbf{x})$.

Want to show $\mathbf{x}' \in B(\mathbf{x}, \delta) \Rightarrow \mathbf{x}' \in B(\mathbf{x}_0, \epsilon)$.

Now, by the triangle inequality,

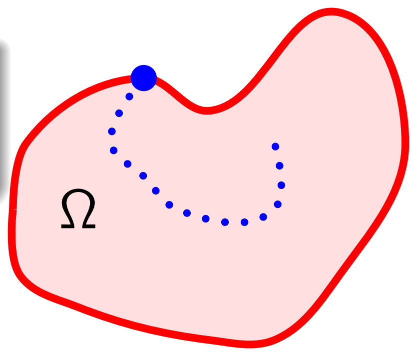
$$d(\mathbf{x}_0, \mathbf{x}') \leq d(\mathbf{x}_0, \mathbf{x}) + d(\mathbf{x}, \mathbf{x}') < r + \delta = r + \epsilon - r = \epsilon.$$

So $\mathbf{x}' \in B(\mathbf{x}_0, \epsilon)$ and hence $B(\mathbf{x}, \delta) \subset B(\mathbf{x}_0, \epsilon)$.

Open and closed sets

Definition

$\mathbf{x}_0$ is a limit point (or accumulation point) of Ω if there is a sequence $\{\mathbf{x}_i\}$ in Ω with limit $\mathbf{x}_0$ and $\mathbf{x}_i \neq \mathbf{x}_0$.



- Every interior point of Ω is a limit point of Ω.
- $\mathbf{x}_0$ is not necessarily in Ω.
- A set is closed $\Leftrightarrow$ it contains all of its limit points.

Definition

- The interior of Ω is the set of all interior points of Ω.
- The boundary of Ω is the set of boundary points of Ω (denoted $\partial\Omega$).
- The closure of Ω is $\Omega \cup \partial\Omega$ (denoted $\bar{\Omega}$).

- Eg, $\bar{\mathbb{Q}} = \mathbb{R}$.
- The interior of Ω is the largest open subset of Ω.
- The closure of Ω is the smallest closed set containing Ω.

Open and closed sets

Tutorial sheet 2 Q4

i) The interesection and union of two open sets is open.

ii) The intersection and union of two closed sets is closed.

What about countable intersections and unions?

Open and closed sets

Tutorial sheet 2 Q5: S closed $\Leftrightarrow$ S contains all of its limit points.

Let's prove "$\Leftarrow$"

Suppose S contains all of its limit points.

[We want to show that S^c is open, ie S^c contains only interior points.]

Let $\mathbf{x} \in S^c$. Now, $\mathbf{x}$ is either an interior point of S^c or a boundary point of S^c.

Assume $\mathbf{x}$ is a boundary point of S^c. That is, all balls around $\mathbf{x}$ contain a point in S^c and a point in $(S^c)^c = S$. So $\mathbf{x}$ is also a boundary point of S.

Next, construct a sequence $\{\mathbf{x}_k\}$ by choosing $\mathbf{x}_k \in B\left(\mathbf{x}, \frac{1}{k}\right) \cap S$

[Note $\mathbf{x}_k \neq \mathbf{x}$ because $\mathbf{x} \notin S$. Also, this makes sense because we have already shown that every ball around $\mathbf{x}$ contains a point in S.]

Since $\frac{1}{k} \to 0$ as $k \to \infty$, it is clear that $\mathbf{x}_k \to \mathbf{x}$, that is, $\mathbf{x}$ is a limit point of S and hence $\mathbf{x} \in S$.

This is a contraction and so S^c is open and hence S is closed.

Limit of a function at a point

Definition

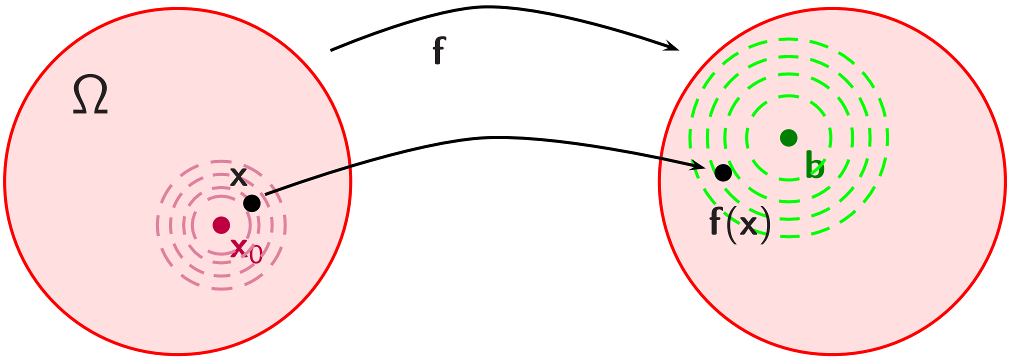
For $\mathbf{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{b}$ means

$$\forall\ \epsilon > 0\ \exists\ \delta > 0 \text{ such that for } \mathbf{x} \in \Omega$$

$$0 < d(\mathbf{x}, \mathbf{x}_0) < \delta \ \Rightarrow\ d(\mathbf{f}(\mathbf{x}), \mathbf{b}) < \epsilon$$

or alternatively

$$\mathbf{x} \in B(\mathbf{x}_0, \delta) \setminus \{\mathbf{x}_0\} \ \Rightarrow\ \mathbf{f}(\mathbf{x}) \in B(\mathbf{b}, \epsilon).$$



Limits example

Show that for $f : \mathbb{R}^2 \setminus \{(0.0)\} \to \mathbb{R}$ with

$$f(x,y) = \frac{x^4 + x^2 + y^2 + y^4}{x^2 + y^2}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 1.$$

$$\begin{aligned} d(f(x,y),1) &= \left| \frac{x^4 + x^2 + y^2 + y^4}{x^2 + y^2} - 1 \right| = \frac{x^4 + y^4}{x^2 + y^2} \\ d((x,y),(0,0)) &= \sqrt{x^2 + y^2} \end{aligned}$$

$$\frac{x^4 + y^4}{x^2 + y^2} \le \frac{x^4 + 2x^2y^2 + y^4}{x^2 + y^2} = \frac{(x^2 + y^2)^2}{x^2 + y^2} = x^2 + y^2 = \left(\sqrt{x^2 + y^2}\right)^2$$

If we choose $\delta = \sqrt{\epsilon}$ then

$$0 < d((x,y),(0,0)) < \delta \ \Rightarrow\ d(f(x,y),1) < \epsilon.$$

Limits

Note: if a limit exists for f, then f approaches that limit along any path.

This can be used to show a limit does not exist, eg,

Show that

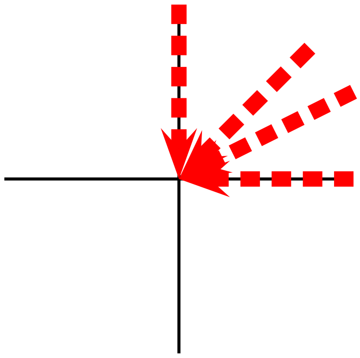
$$\lim_{(x,y)\to(0,0)} f(x,y)$$

does not exist for

- $f(x,y) = \dfrac{xy}{x^2+y^2}$
- $f(x,y) = \dfrac{x^2 y}{x^4+y^2}$

Limits

$$f(x,y) = \frac{xy}{x^2+y^2}, \qquad f : \mathbb{R}^2 \setminus \mathbf{0} \to \mathbb{R}.$$



Approach along positive x-axis (ie $y=0$)

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0^+} \frac{0}{x^2} = 0.$$

Similarly for approaching along the positive y-axis.

Approach along the line $y=x$ in the first quadrant

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0^+} \frac{x\ x}{x^2+x^2} = \lim_{x\to 0^+} \frac{1}{2} = \frac{1}{2}.$$

Approach along the line $y=mx$ in the first quadrant

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0^+} \frac{x\ mx}{x^2+m^2x^2} = \lim_{x\to 0^+} \frac{m}{m^2+1} = \frac{m}{m^2+1}.$$

A different limit is reached approaching $(0,0)$ along different straight lines. Hence the limit of $f(x,y)$ as $(x,y)\to(0,0)$ does not exist.

Limits

$$f(x,y) = \frac{x^2y}{x^4+y^2}, \qquad f : \mathbb{R}^2 \setminus \mathbf{0} \to \mathbb{R}.$$

$$\begin{aligned} f(x,mx) &= \frac{x^2\ mx}{x^4+m^2x^2} = \frac{mx^3}{x^4+m^2x^2} \\ &= \frac{mx}{x^2+m^2} \to 0 \quad \text{as} \quad x \to 0. \end{aligned}$$

$$f(x,ax^2) = \frac{x^2\ ax^2}{x^4+a^2x^4} = \frac{a}{1+a^2}.$$

Hence a different limit is attained by approaching along different parabolas and so $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Limits

For $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ it is sufficient to consider the limits of the components of $\mathbf{f}$.

Example

$\mathbf{f} : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$

$$f(x,y) = \left(\frac{x^3}{x^2+y^2}, \frac{x^2+y^2+x^2y^2}{x^2+y^2} \right)$$

If we are given that

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} &= 0 \\ \lim_{(x,y)\to(0,0)} \frac{x^2+y^2+x^2y^2}{x^2+y^2} &= 1 \end{aligned}$$

then

$$\lim_{(x,y)\to(0,0)} f(x,y) = (0,1).$$

Algebra of limits

Theorem

For

$$f : \mathbb{R}^n \to \mathbb{R} \quad \text{and} \quad g : \mathbb{R}^n \to \mathbb{R}$$

with

$$\lim_{\mathbf{x}\to\mathbf{x_0}} f(\mathbf{x}) = a \quad \text{and} \quad \lim_{\mathbf{x}\to\mathbf{x_0}} g(\mathbf{x}) = b.$$

Then

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{x_0}} (f+g)(\mathbf{x}) &= a+b \\
\lim_{\mathbf{x}\to\mathbf{x_0}} (fg)(\mathbf{x}) &= ab \\
\lim_{\mathbf{x}\to\mathbf{x_0}} (f/g)(\mathbf{x}) &= a/b \qquad \text{provided } b \neq 0.
\end{aligned}$$

For $f : \mathbb{R}^m \to \mathbb{R}$, $g : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, is it true that $\lim_{\mathbf{x}\to\mathbf{x_0}} (f \circ g)(\mathbf{x}) = f(g(\mathbf{x}_0))$?
For this we need continuity.

Algebra of limits

If we can prove

$$\lim_{\mathbf{x}\to\mathbf{a}} c = c \quad \text{and} \quad \lim_{\mathbf{x}\to\mathbf{a}} x_i = a_i$$

then we can use the algebra of limits to find limits for rational functions.

To prove $\lim_{\mathbf{x}\to\mathbf{a}} c = c$, for each $\epsilon > 0$ choose $\delta = 1$.

To prove $\lim_{\mathbf{x}\to\mathbf{a}} x_i = a_i$, for each $\epsilon > 0$ choose $\delta = \epsilon$.

Continuity

Definition

$\mathbf{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\mathbf{x}_0 \in \Omega$ means either

i) $\mathbf{x}_0$ is a limit point of Ω, $\lim_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x})$ exists and equals $\mathbf{f}(\mathbf{x}_0)$, or

ii) $\mathbf{x}_0$ is not a limit point of Ω.

$\mathbf{f}$ is continuous on Ω if it is continuous at each point of Ω.

Continuity

Theorem

Suppose that $\mathbf{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ *and* $\mathbf{x}_0 \in \Omega$. *The following are equivalent.*

i) $\mathbf{f}$ *is continuous at* $\mathbf{x}_0 \in \Omega$.

ii) $\forall \epsilon > 0 \ \exists \delta > 0$ *such that for* $\mathbf{x} \in \Omega$, $d(\mathbf{x}, \mathbf{x}_0) < \delta \ \Rightarrow \ d(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{x}_0)) < \epsilon$.
[Ie $\mathbf{x} \in B(\mathbf{x}_0, \delta) \Rightarrow \mathbf{f}(\mathbf{x}) \in B(\mathbf{f}(\mathbf{x}_0), \epsilon)$.*]*

iii) $\forall$ *sequences* $\{\mathbf{x}_k\}$ *in* Ω *with limit* $\mathbf{x}_0$, $\{\mathbf{f}(\mathbf{x}_k)\}$ *converges to* $\mathbf{f}(\mathbf{x}_0)$.

iv) $\mathbf{f}(\mathbf{x}_0)$ *is an interior point of* $\mathbf{f}(\Omega) \ \Rightarrow \ \mathbf{x}_0$ *is an interior point of* Ω.

Theorem

Suppose that $\mathbf{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$. *The following two statements are equivalent.*

- $\mathbf{f}$ *is continuous on* Ω.
- U *is open in* $\mathbb{R}^m \ \Rightarrow \ \mathbf{f}^{-1}(U)$ *is open in* $\mathbb{R}^n$.

The preimage $f^{-1}(U) = \{\mathbf{y} \in \mathbb{R}^n : \mathbf{f}(\mathbf{y}) \in U\}$.

The second statement is an alternative definition of continuity. (Tutorial sheet 2.)

Continuity (iii) $\Rightarrow$ (i) $[\neg(\text{i}) \Rightarrow \neg(\text{iii})]$

$$\neg\Big(\forall\epsilon > 0\ \exists\delta > 0\ \forall\mathbf{x} \in B(\mathbf{x}_0, \delta)\ \Rightarrow\ \mathbf{f}(\mathbf{x}) \in B(\mathbf{f}(\mathbf{x}_0, \epsilon)\Big)$$

$$\exists\epsilon > 0\ \neg\Big(\exists\delta > 0\ \forall\mathbf{x} \in B(\mathbf{x}_0, \delta)\ \Rightarrow\ \mathbf{f}(\mathbf{x}) \in B(\mathbf{f}(\mathbf{x}_0, \epsilon)\Big)$$

$$\exists\epsilon > 0\ \forall\delta > 0\ \neg\Big(\forall\mathbf{x} \in B(\mathbf{x}_0, \delta)\ \Rightarrow\ \mathbf{f}(\mathbf{x}) \in B(\mathbf{f}(\mathbf{x}_0, \epsilon)\Big)$$

$$\exists\epsilon > 0\ \forall\delta > 0\ \exists\mathbf{x} \in B(\mathbf{x}_0, \delta)\ \mathbf{f}(\mathbf{x}) \notin B(\mathbf{f}(\mathbf{x}_0, \epsilon)$$

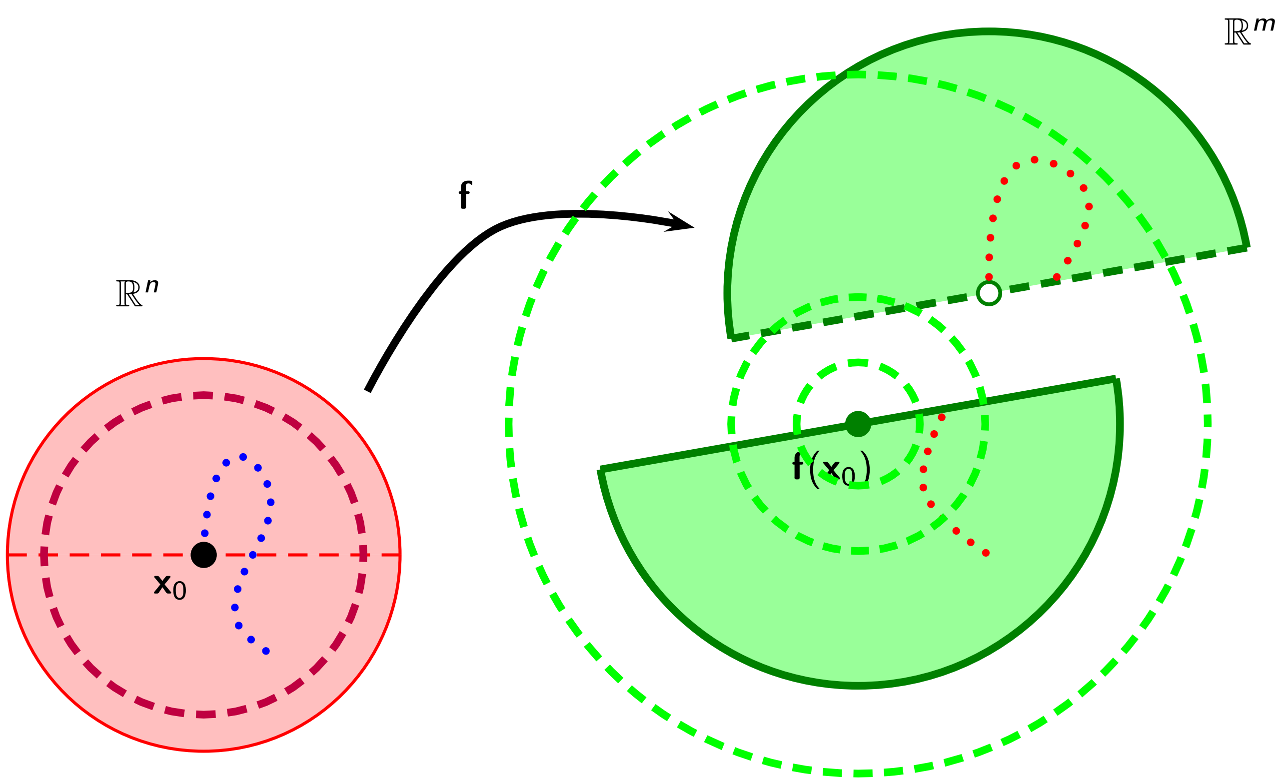
That is,

$$\exists\epsilon > 0 \text{ such that } \forall\delta > 0\ \exists\mathbf{x} \in B(\mathbf{x}_0, \delta) \text{ such that } \mathbf{f}(\mathbf{x}) \notin B(\mathbf{f}(\mathbf{x}_0, \epsilon) \qquad (*)$$

- Choose an ϵ verifying $(*)$.
- In each ball $B(\mathbf{x}_0, \frac{1}{k})$ choose $\mathbf{x}_k \in B(\mathbf{x}_0, \frac{1}{k}) \cap \Omega$ such that $\mathbf{f}(\mathbf{x}_k) \notin B(\mathbf{f}(\mathbf{x}_0), \epsilon)$. [Can do this because of $(*)$.]

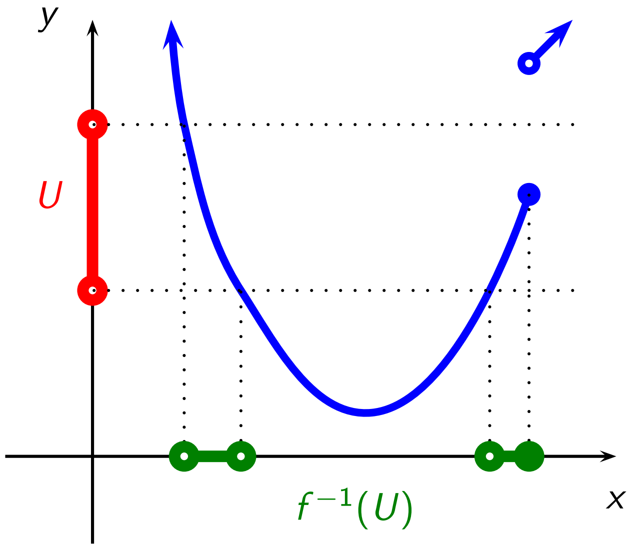
This is a sequence with $\mathbf{x}_k \to \mathbf{x}_0$ but $\mathbf{f}(\mathbf{x}_k) \not\to \mathbf{f}(\mathbf{x}_0)$.

Continuity (iii) $\Rightarrow$ (i) $[\neg(\text{i}) \Rightarrow \neg(\text{iii})]$



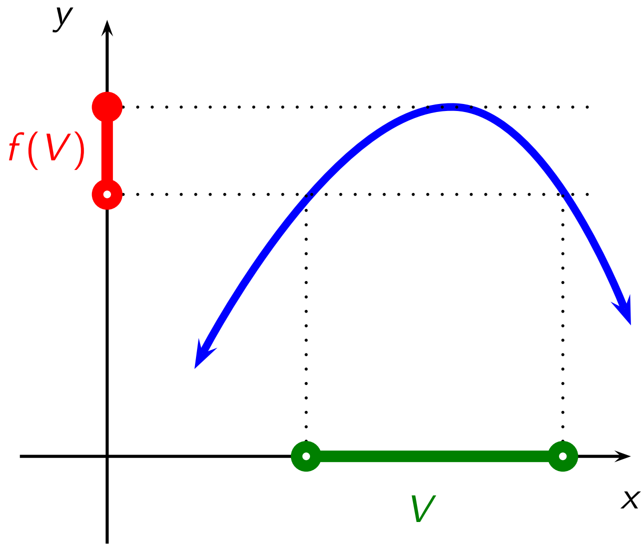
Continuity — alternative definition

Tutorial sheet 2 Q15 gives another definition of continuity.



f is not continuous. U is open but $f^{-1}(U)$ is not open.

Note: a continuous function can map open sets to closed sets.



f is continuous. V is open but $f(V)$ is not open.

Algebra of continuous functions

Theorem

A function $\mathbf{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ *is continuous on* Ω *if and only if its component functions are continuous.*

Theorem (Algebra of continuous functions)

For two functions continuous on Ω

$$f : \Omega \subset \mathbb{R}^n \to \mathbb{R} \quad \textit{and} \quad g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$$

$f + g$, fg *and* f/g *are continuous. [The domain of* f/g *must exclude points where* $g(\mathbf{x}) = 0$.*]*

Note also that for

$$f : \mathbb{R}^m \to \mathbb{R} \quad \text{and} \quad g : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$$

$f \circ g$ is continuous where it makes sense.

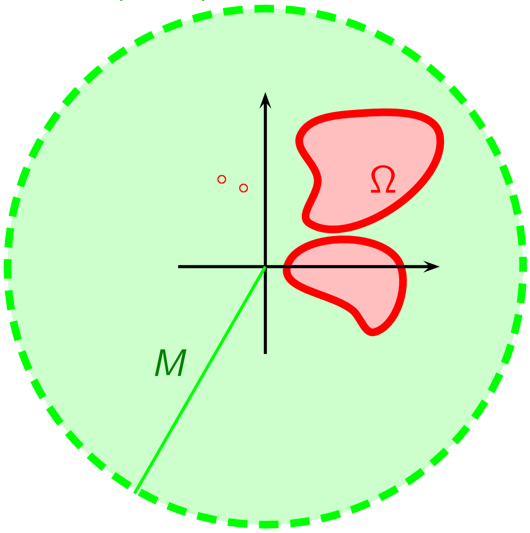
Compact and connected sets

Definition

A set $\Omega \subset R^n$ is bounded if there is an M such that $d(\mathbf{x}, \mathbf{0}) \leq M$ for all $\mathbf{x} \in \Omega$.

Example:

$\Omega \subset B(\mathbf{0}, M)$



Example:

$B(\mathbf{x}_0, \epsilon)$ is bounded $\forall \mathbf{x}_0 \in \mathbb{R}^n$ and $\forall \epsilon > 0$.

Proof:

for $\mathbf{x} \in B(\mathbf{x}_0, \epsilon)$ choose $M = \epsilon + d(\mathbf{x}_0, \mathbf{0})$.

$$\begin{aligned} d(\mathbf{x}, \mathbf{0}) &\leq d(\mathbf{x}, \mathbf{x}_0) + d(\mathbf{x}_0, \mathbf{0}) \\ &< \epsilon + d(\mathbf{x}_0, \mathbf{0}) \\ &= M. \end{aligned}$$

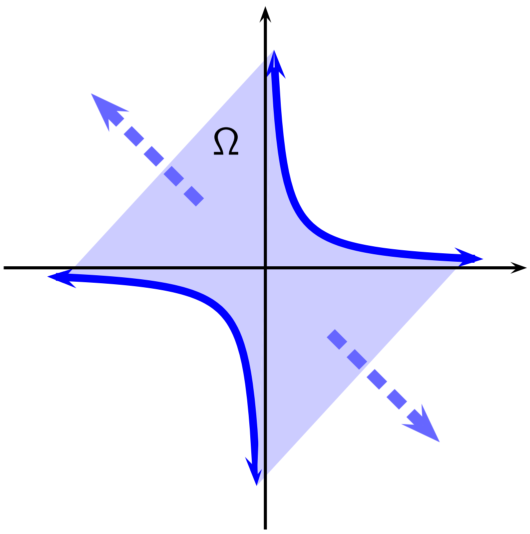
Compact and connected sets

Example:

The set

$$\Omega = \{(x, y) \in \mathbb{R}^2 \ : \ xy \leq 1\}$$

is not bounded.



Suppose $\Omega \subset B(\mathbf{0}, M)$.

Now, $(M+1, 0) \in \Omega$ since $(M+1).0 = 0 \leq 1$. But

$$d\Big((M+1, 0), (0, 0)\Big) = M + 1 > M$$

so $(M+1, 0) \notin B(\mathbf{0}, M)$ and hence $(M+1, 0) \notin \Omega$.

This is a contradiction and so $\Omega \not\subset B(\mathbf{0}, M)$. Hence Ω is not bounded.

Monotone convergence theorem

Theorem (Monotone convergence theorem)

A bounded monotone sequence in $\mathbb{R}$ converges to a limit in $\mathbb{R}$.

This relies on the existence of a least upper bound for bounded set in $\mathbb{R}$. (Note that $\mathbb{Q}$ does not have this property.)

Lemma

Every bounded sequence in $\mathbb{R}$ has a monotone subsequence.

Theorem

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence with a limit in $\mathbb{R}$.

Bolzano-Weierstrass theorem

Theorem

For $\Omega \subset \mathbb{R}^n$, the following are equivalent.

(i) *Ω is closed and bounded.*

(ii) *Every sequence in Ω has a subsequence that converges to an element of Ω.*

A third equivalent statement that is beyond the scope of this course is given by the Heine-Borel theorem.

(iii) Whenever the union of a collection of open sets contains Ω there is always a finite sub-collection thats union also contains Ω.

Bolzano-Weierstrass theorem, proof of (i) $\Rightarrow$ (ii)

Suppose Ω is closed and bounded and let $\{\mathbf{x}_k\}$ be a sequence in Ω.

- The first components form a bounded sequence in $\mathbb{R}$.
- Choose a subsequence for which the first components converge (to $x_{0,1}$).
- Choose a subsequence of this subsequence for which the second components converge (to $x_{0,2}$).
- Repeat for each component.
- We now have a subsequence $\{\mathbf{x}_{k_l}\}$ that converges to $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,n})$. But is $\mathbf{x}_0 \in \Omega$?
- Suppose $\mathbf{x}_0 \in \Omega^c$ which is open because Ω is closed.
- So, $\exists \epsilon > 0$ such that $B(\mathbf{x}_0, \epsilon) \subset \Omega^c$ and so no element of $\{\mathbf{x}_{k_l}\}$ is in $B(\mathbf{x}_0, \epsilon)$.
- This contradicts $\mathbf{x}_{k_l} \to \mathbf{x}_0$. So $\mathbf{x}_0 \in \Omega$.

That is, $\{\mathbf{x}_k\}$ has a convergent subsequence with limit in Ω.

$(1,0,\frac{1}{2})$, $(-1,\frac{1}{2},-\frac{1}{2})$,
$(\frac{1}{2},0,0)$, $(-1,0,-\frac{3}{4})$,
$(\frac{1}{3},\frac{2}{3},\frac{7}{8})$, $(-1,0,-\frac{7}{8})$,
$(\frac{1}{4},0,\frac{15}{16})$, $(-1,\frac{3}{4},-\frac{15}{16})$,
$(\frac{1}{5},0,\frac{31}{32})$, $(-1,0,-\frac{31}{32})$,
$(\frac{1}{6},\frac{4}{5},0)$, $(-1,0,-\frac{63}{64})$,
$(\frac{1}{7},0,\frac{127}{128})$,
$(-1,\frac{5}{6},-\frac{127}{128})$,
$(\frac{1}{8},0,\frac{255}{256})$,
$(-1,0,-\frac{255}{256})$,
$(\frac{1}{9},\frac{6}{7},\frac{511}{512})$,
$(-1,0,-\frac{-511}{512})$, ...
$(1,0,\frac{1}{2})$, $(\quad,\quad,\quad)$,
$(1\;0\;0)$, $($

$(\frac{1}{3},\frac{2}{3},\frac{7}{8})$, $(\quad,\quad,\quad)$,

Bolzano-Weierstrass theorem, $\neg$(i) $\Rightarrow$ $\neg$(ii), ie (ii) $\Rightarrow$ (i)

- If Ω is not bounded,
  - choose a sequence $\{\mathbf{x}_k\}$ with $d(\mathbf{x}_k, \mathbf{0}) > k$.
  - This sequence has no convergent subsequence.
- If Ω is not closed,
  - it has a boundary point $\mathbf{x}_0$ not in Ω.
  - Construct a sequence $\{\mathbf{x}_k\}$ by choosing $\mathbf{x}_k$ to be any point in $B(\mathbf{x}_0, \frac{1}{k}) \cap \Omega$ which must be non-empty.
  - Clearly $\{\mathbf{x}_k\}$ converges with limit $\mathbf{x}_0$ and hence any subsequence converges with limit $\mathbf{x}_0$.

That is, we have constructed a sequence with no convergent subsequence with limit in Ω.

Compact sets

Definition

A set Ω is compact if it satisfies either property from the Bolzano-Weierstrass theorem.

Examples:

$\emptyset$	compact
$\mathbb{R}$	not compact
$(0,1)$	not compact
$[0,1]$	compact
$[0,1]\cup[3,4]$	compact
$[0,1]\times[0,1]$	compact
S^2 (the 2-sphere)	compact
Cantor set	compact

Path connected sets

Definition

A continuous path between $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$ is a function $\phi:[0,1]\to\mathbb{R}^n$ such that ϕ is continuous and $\phi(0)=\mathbf{x}$, $\phi(1)=\mathbf{y}$.

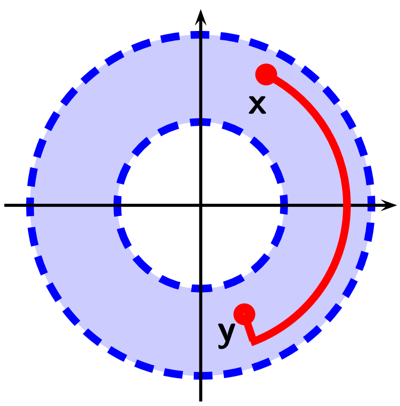
Definition

A set $\Omega\subset\mathbb{R}^n$ is said to be path connected if, given any $\mathbf{x},\mathbf{y}\in\Omega$, there is a continuous path between them that lies entirely in Ω (ie $\forall t\in[0,1]\ \phi(t)\in\Omega$).

Example: For $\Omega=\{(x,y)\ :\ 1<x^2+y^2<4\}$ show that Ω is path connected.

$$\phi(t)=\begin{cases}\Big(r_1\cos\theta(t),r_1\sin\theta(t)\Big) & \text{for } 0\le t\le\frac{1}{2}\\ \Big(r(t)\cos\theta_2,r(t)\sin\theta_2\Big) & \text{for } \frac{1}{2}<t\le 1\end{cases}$$

with $\theta(t)=\theta_1+2(\theta_2-\theta_1)t,\ r(t)=r_1+2(r_2-r_1)(t-\frac{1}{2})$ and $\mathbf{x}=(r_1\cos\theta_1,r_1\sin\theta_1)$, $\mathbf{y}=(r_2\cos\theta_2,r_2\sin\theta_2)$.



Path connected sets

Show that $\Omega = \{(x, y) \ : \ xy > 1\}$ is not path connected.

Suppose that Ω is path connected. So there must be a continuous path between $(-2,-2)$ and $(2,2)$ that lies entirely in Ω. That is, there is a continuous function $\phi : [0,1] \to \mathbb{R}^2$ with $\phi(0) = (-2,-2)$, $\phi(1) = (2,2)$, $\phi(t) \in \Omega \ \forall t \in [0,1]$.

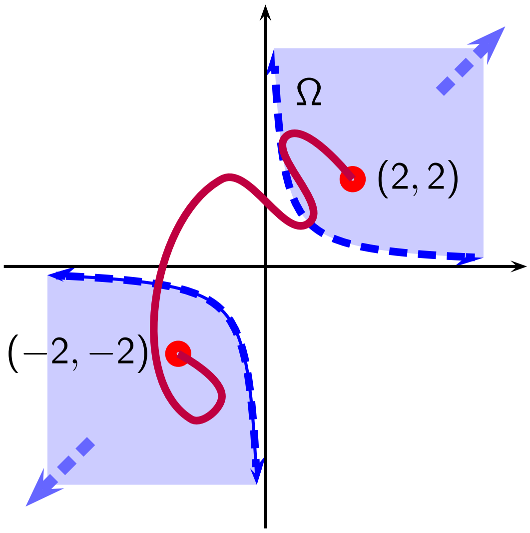
The first component of ϕ

$$\text{ie} \quad \phi_1 : [0,1] \to \mathbb{R}$$

must be continuous on $[0,1]$ with $\phi_1(0) = -2$ and $\phi_1(1) = 2$. So the Intermediate Value Theorem says

$$\exists \ c \in [0,1] \text{ such that } \phi_1(c) = 0$$

and hence $\phi(c) \notin \Omega$. This is a contradiction and hence Ω can not be path connected.



Big theorems

Theorem

Let $\mathbf{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be continuous. Then

(i) *$K \subset \Omega$ and K is compact $\Rightarrow$ $\mathbf{f}(K)$ is compact.*

(ii) *$B \subset \Omega$ and B is path connected $\Rightarrow$ $\mathbf{f}(B)$ is path connected.*

Proof (i):

Let $\mathbf{f}$ be continuous, K compact and $\{\mathbf{y}_k\}$ be a sequence in $\mathbf{f}(K)$.

So there is $\{\mathbf{x}_k\}$ such that $\mathbf{x}_k \in K$ and $\mathbf{y}_k = \mathbf{f}(\mathbf{x}_k)$.

K compact $\Rightarrow$ there is a convergent subsequence $\{\mathbf{x}_{k_l}\}$ with limit $\mathbf{x} \in K$.

$\mathbf{f}$ continuous $\Rightarrow$ $\{\mathbf{f}(\mathbf{x}_{k_l})\}$, that is, $\{\mathbf{y}_{k_l}\}$ is a convergent subsequence of $\{\mathbf{y}_k\}$ with limit $\mathbf{f}(\mathbf{x}) = \mathbf{y} \in \mathbf{f}(K)$.

That is, we have shown that for any sequence in $\mathbf{f}(K)$, there exists a convergent subsequence with limit in $\mathbf{f}(K)$ and hence $\mathbf{f}(K)$ is compact.

Big theorems

Proof (ii):

Let $\mathbf{f}$ be continuous, B path connected and $\mathbf{y}_1, \mathbf{y}_2 \in \mathbf{f}(B)$.

So there are $\mathbf{x}_1, \mathbf{x}_2 \in B$ such that $\mathbf{y}_1 = \mathbf{f}(\mathbf{x}_1)$ and $\mathbf{y}_2 = \mathbf{f}(\mathbf{x}_2)$.

B is path connected means there is a continuous function $\phi : [0,1] \to B$ such that

$$\phi(0) = \mathbf{x}_1, \;\; \phi(1) = \mathbf{x}_2 \text{ and } \phi(t) \in B \;\forall t \in [0,1].$$

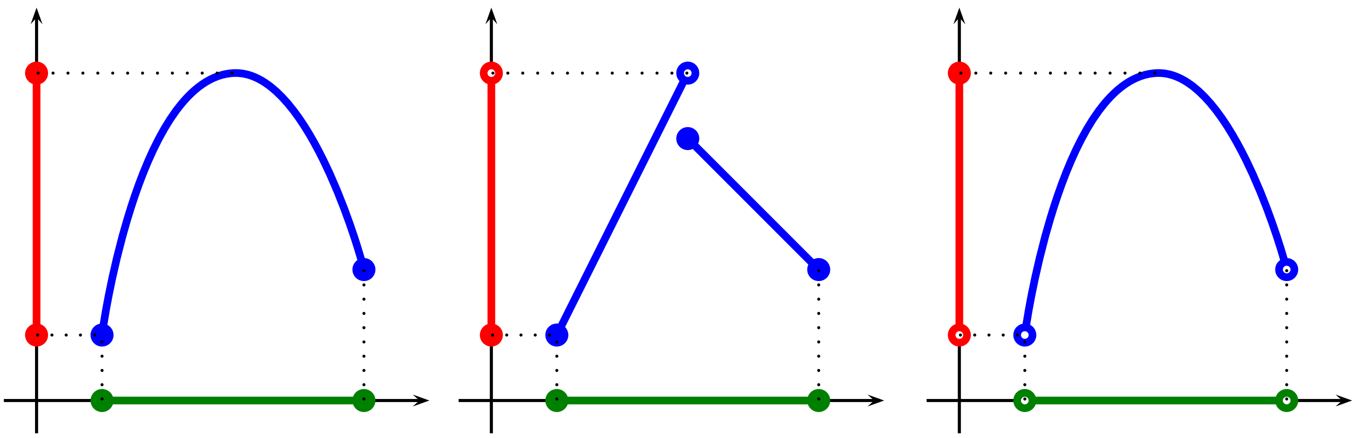
Since $\mathbf{f}$ is continuous, $\mathbf{f} \circ \phi : [0,1] \to \mathbf{f}(B)$ is continuous with

$$\begin{aligned}
(\mathbf{f} \circ \phi)(0) &= \mathbf{f}(\phi(0)) = \mathbf{f}(\mathbf{x}_1) = \mathbf{y}_1 \\
(\mathbf{f} \circ \phi)(1) &= \mathbf{f}(\phi(1)) = \mathbf{f}(\mathbf{x}_2) = \mathbf{y}_2 \\
(\mathbf{f} \circ \phi)(t) &\in \mathbf{f}(B) \text{ for } t \in [0,1].
\end{aligned}$$

That is, $\mathbf{f} \circ \phi$ is a continuous path between $\mathbf{y}_1$ and $\mathbf{y}_2$ contained in $\mathbf{f}(B)$.

Hence $\mathbf{f}(B)$ is path connected.

Min/max theorem for $f : \mathbb{R} \to \mathbb{R}$

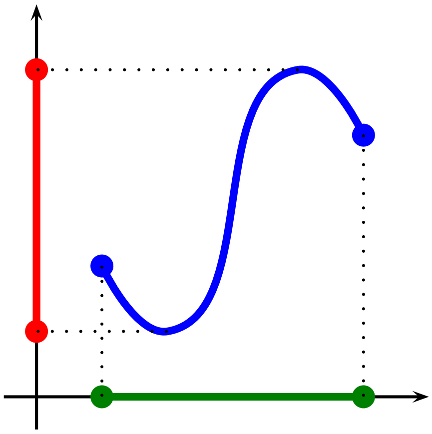


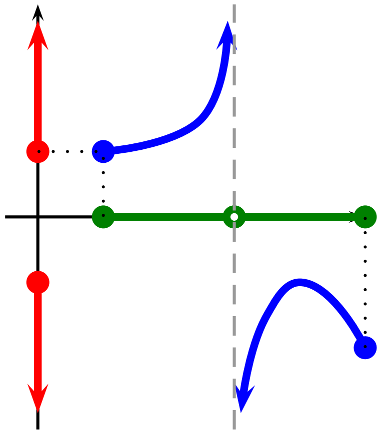
For $f : K \subset \mathbb{R} \to \mathbb{R}$ continuous on a compact set K, maximum and minimum values are attained.

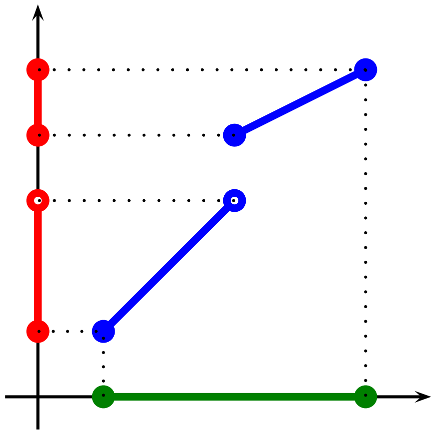
For $f : K \subset \mathbb{R} \to \mathbb{R}$ not continuous on a compact set K, maximum and minimum values may or may not be attained.

For $f : \Omega \subset \mathbb{R} \to \mathbb{R}$ continuous on a non-compact set Ω, maximum and minimum values may or may not be attained.

Intermediate Value Theorem for $f : \mathbb{R} \to \mathbb{R}$







For $f : B \subset \mathbb{R} \to \mathbb{R}$ continuous on a path connected set B, $f(B)$ is path connected.

For $f : B \subset \mathbb{R} \to \mathbb{R}$ continuous on a not path connected set B, $f(B)$ is not necessarily path connected.

For $f : B \subset \mathbb{R} \to \mathbb{R}$ not continuous on a path connected set B, $f(B)$ is not necessarily path connected.

Big theorems

Consider

$$S_1 = \{(x, y) \ : \ x^2 + y^2 \leq 1\} \qquad S_2 = \{(x, y) \ : \ x^2 + y^2 < 1\}$$

Is there a continuous function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ such that

1. $\mathbf{f}(S_1) = S_2$?
2. $\mathbf{f}(S_2) = S_1$?
3. $\mathbf{f}(\mathbb{R}^2) = S_2$?
4. $\mathbf{f}(\mathbb{R}^2) = S_1$?
5. $\mathbf{f}(S_2) = \mathbb{R}^2$?
6. $\mathbf{f}(S_1) = \mathbb{R}^2$?

Big theorems

Consider $S_1 = \{(x,y) : x^2+y^2 \le 1\}$ and $S_2 = \{(x,y) : x^2+y^2 < 1\}$.
S_1, S_2 and $\mathbb{R}^2$ are path connected but only S_1 is compact.

1. S_1 is compact and S_2 is not. So there can not be a continuous function $\mathbf{f}_1$ with $\mathbf{f}_1(S_1) = S_2$.
2. Consider the function $\mathbf{f}_2 : \mathbb{R}^2 \to \mathbb{R}^2$ described in terms of polar coordinates by
$$(r,\theta) \to \begin{cases} (2r,\theta) & \text{for } r < \frac{1}{2} \\ (1,\theta) & \text{for } r \ge \frac{1}{2}. \end{cases}$$
This is continuous and $\mathbf{f}_2(S_2) = S_1$.
3. Consider the function $\mathbf{f}_3 : \mathbb{R}^2 \to \mathbb{R}^2$ described in polar coordinates by
$$(r,\theta) \to \left(\frac{2}{\pi}\tan^{-1} r, \theta\right).$$
This is continuous and $\mathbf{f}_3(\mathbb{R}^2) = S_2$.
4. $\mathbf{f}_2 \circ \mathbf{f}_3$ is a continuous function that maps $\mathbb{R}^2$ to S_1.
5. $\mathbf{f}_5 = \mathbf{f}_3^{-1}$ is a continuous function with $f(S_2) = \mathbb{R}^2$.
6. S_1 is compact and $\mathbb{R}^2$ is not. So there can not be a continuous function $\mathbf{f}_6$ with $\mathbf{f}_6(\mathbb{R}^2) = S_1$.

Example

Prove that if the temperature is above 0 somewhere on the Earth's surface and below 0 somewhere else, then there must be a third point where it is exactly 0.

The surface of the Earth S^2 is compact and path connected and (assume) that the tempature $T : S^2 \to \mathbb{R}$ is continuous.

So the image of S^2 under T, $T(S^2)$, must compact and path connected. That is $T(S^2)$ is a closed bounded interval $[a,b]$.

There is a point $\mathbf{x}$ where $T(\mathbf{x}) < 0$ and $\mathbf{y}$ where $T(\mathbf{y}) > 0$. That is, $[a,b]$ contains both positive and negative values and hence $0 \in [a,b]$.

Hence there is $\mathbf{u} \in S^2$ such that $T(\mathbf{u}) = 0$.