MATH2111 Higher Several Variable Calculus*
Tutorial Problems

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Contents

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*http://web.maths.unsw.edu.au/~potapov/2111_2015/

1 Curves and Surfaces

[M] – Maple/Gnuplot; [A] – additional/optional problems; [H] – harder problems.

1.1 Curves in $\mathbb{R}^n$

1: Sketch the curves $(x, y) = (t, t)$ and $(x, y) = (t^2, t+2)$ for $t \in \mathbb{R}$, and find the two points where they intersect.

2: Sketch the projections of the following curves onto the plane $z = 0$ and onto the plane $y = 0$.

i) $(x, y, z) = (\cos 2t, \sin 2t, \sin t)$, $t \in \mathbb{R}$.

ii) $(x, y, z) = (\cos t, \sin t, \sin 3t)$, $t \in \mathbb{R}$. (Rotate the image until it looks like the ABC logo).

[M] **3**: Sketch the curves in Q2 by the command `spacecurve` in Maple; `plot3` in Matlab; or `splot` in Gnuplot. Rotate the image to see what the curve looks like from different viewpoints.

4: Find the unit tangent vector to the parametrised curve $\boldsymbol{r}(t)$ at $t = a$ and write down a parametric equation for the tangent line to the curve at a.

i) $\boldsymbol{r}(t) = 3\cos t\,\boldsymbol{i} + 3\sin t\,\boldsymbol{j} + 4t\boldsymbol{k}$, $a = \pi/4$

ii) $\boldsymbol{r}(t) = t\,\boldsymbol{i} + t^2\,\boldsymbol{j} + t^3\,\boldsymbol{k}$, $a = 1$.

5: Consider the two curves given in parametric form by $\boldsymbol{r}(t) = (t^2 - t, t^2 + t)$ and $\boldsymbol{r}(t) = (t + t^2, t - t^2)$ for $t \in \mathbb{R}$.

i) Find the *two* points of intersection of the curves.

ii) Find the angle between the two curves at each point of intersection.

iii) Find all points on the curves where the tangent is parallel to $\boldsymbol{i}$.

iv) Find all points on the curves where the tangent is parallel to $\boldsymbol{j}$.

v) For $-2 \le t \le 2$, sketch both curves on the same diagram. Show clearly all the points and angles you have found.

6: Sketch the curve given parametrically by $(x, y) = (t^3, t^5)$, $t \in \mathbb{R}$. Show that this parametrisation does not give a tangent vector for the curve at $(0, 0)$. Find a parametrisation of this curve which does give a tangent vector for the curve at $(0, 0)$.

[A] **7**: Suppose that $\boldsymbol{f} : \mathbb{R} \to \mathbb{R}^3$, $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^3$ and $\lambda : \mathbb{R} \to \mathbb{R}$. Prove that

i) $\dfrac{d}{dt}(\lambda \boldsymbol{f}) = \lambda \dfrac{d\boldsymbol{f}}{dt} + \dfrac{d\lambda}{dt}\boldsymbol{f}$.

ii) $\dfrac{d}{dt}(\boldsymbol{f} \cdot \boldsymbol{g}) = \dfrac{d\boldsymbol{f}}{dt} \cdot \boldsymbol{g} + \boldsymbol{f} \cdot \dfrac{d\boldsymbol{g}}{dt}$.

iii) $\dfrac{d}{dt}(\boldsymbol{f} \times \boldsymbol{g}) = \dfrac{d\boldsymbol{f}}{dt} \times \boldsymbol{g} + \boldsymbol{f} \times \dfrac{d\boldsymbol{g}}{dt}$.

8: Suppose $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are three differentiable functions from $\mathbb{R}$ to $\mathbb{R}^3$ such that for every $t \in \mathbb{R}$ the vectors $\boldsymbol{u}(t)$, $\boldsymbol{v}(t)$, $\boldsymbol{w}(t)$ form an orthonormal basis in $\mathbb{R}^3$.

i) Prove that $\boldsymbol{u}'(t) \perp \boldsymbol{u}(t)$ and $\boldsymbol{u}'(t) \cdot \boldsymbol{v}(t) = -\boldsymbol{u}(t) \cdot \boldsymbol{v}'(t)$ for all t.

ii) Suppose that $\boldsymbol{r}(t) = x(t)\boldsymbol{u}(t) + y(t)\boldsymbol{v}(t) + z(t)\boldsymbol{w}(t)$ for some three functions x, y, z from $\mathbb{R} \to \mathbb{R}$. Show that $x(t) = \boldsymbol{r}(t) \cdot \boldsymbol{u}(t)$ and $y(t) = \boldsymbol{r}(t) \cdot \boldsymbol{v}(t)$ and $z(t) = \boldsymbol{r}(t) \cdot \boldsymbol{w}(t)$ for all t.

9: A particle moves in a plane such that its position at time t is given by $\boldsymbol{r}(t) = (3t^2, t^3 - 9t)$.

i) Find all positions at which the velocity of the particle is perpendicular to its acceleration.

ii) Show that there are no positions where the velocity of the particle is parallel to its acceleration.

10: i) Show that for a particle moving with velocity $\boldsymbol{v}(t)$, if $\boldsymbol{v}(t) \cdot \boldsymbol{v}'(t) = 0$ for all t then the speed v is constant.

ii) A particle of mass m with position vector $\boldsymbol{r}(t)$ at time t is acted on by a total force

$$\boldsymbol{F}(t) = \lambda \boldsymbol{r}(t) \times \boldsymbol{v}(t),$$

where λ is a constant and $\boldsymbol{v}(t)$ is the velocity of the particle. Show that the speed v of the particle is constant. (Note that Newton's second law of motion in its vector form is $\boldsymbol{F} = m\boldsymbol{a}$.)

11: At time t a particle is at position $\boldsymbol{r}(t)$.

i) Show that

$$\boldsymbol{r} \cdot \frac{d\boldsymbol{r}}{dt} = \frac{d}{dt}\left(\frac{1}{2}r^2\right),$$

where r is the distance of the particle from the origin.

ii) Show $\dfrac{d(v^2)}{dt} = 2\dfrac{d^2\boldsymbol{r}}{dt^2} \cdot \dfrac{d\boldsymbol{r}}{dt}$

1.2 Surfaces in $\mathbb{R}^3$

[M] **12**: Use Maple/Gnuplot to plot the following functions $f : \mathbb{R}^2 \to \mathbb{R}$.

i) $f(x, y) = \cos(x + y)$.

ii) $f(x, y) = \sin(\sqrt{x^2 + y^2})$.

iii) $f(x, y) = x^3 - x^2y$.

iv) $f(x, y) = x^4 - 2x^2y$.

13: i) Sketch level curves $f = \pm 1$ for the functions (i) and (ii) of Q12.

ii) Sketch the section of the function (i) of Q12 by the planes $x = \pm\frac{\pi}{4}; \pm\frac{\pi}{2}$.

iii) Sketch the section of function (ii) of Q12 by the plane $y = x\tan\theta$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

[M] **14**: Sketch the graph of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = y^3 - y - 2x^2$. Use the `plot3d` command in Maple (or `surf` in Matlab; or `splot` of Gnuplot) to sketch the graph for $|x| \le 1.2$, $|y| \le 1.5$. Use the Maple command `contourplot` (or the Matlab command `contour`; or `splot` with `set contour` in Gnuplot, see `help contour` in Gnuplot) to look at some of the contours of f.

[M] **15**: Apply the Maple command `contourplot` (or the Matlab command `contour`; or `splot` with `set contour` in Gnuplot, see `help contour` in Gnuplot) to the function $f(x, y) = x/(1 + x^2 + y^2)$ for $|x| \le 3$, $|y| \le 2$ and then use these contours to sketch the surface. (Note that the contours in the Maple sketch are not labelled with the corresponding values of f, but by default their colour shades from yellow to red as the value of f increases.) Use `plot3d` (or `splot` of Gnuplot) to check your sketch.

[M] **16**: On separate diagrams, sketch the surfaces in $\mathbb{R}^3$ defined by the following equations:

i) $z = x^2 + y^2$;

ii) $2z^2 = x^2 + y^2$;

iii) $x^2 + y^2 + z^2 = 9$;

iv) $x^2 + y^2 = 4$;

v) $x^2 + y^2 - z^2 = 1$;

vi) $x^2 - y^2 - z^2 = 1$.

17: For each function of Q16, sketch the sections of the graph of the function by the planes $x = \alpha$, $y = \alpha$ and $z = \alpha$, where $\alpha = -1, 0, 1$.

18: Consider the region above the cone $z^2 = x^2 + y^2$, $z \ge 0$, and inside the sphere $x^2 + y^2 + z^2 = 2az$, with $a > 0$.

i) Sketch the section of this region by the plane $x = 0$.

ii) Describe the curve of intersection of these surfaces.

iii) What is the projection of the region on the x, y plane?

19: i) Parametrise the curve of intersection of two cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

ii) What is the curve of intersection.

iii) Find the projection of the curve of intersection onto the plane $z = 0$.

20: Let

$$S = \{\, (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z - 1)^2 = 1 \,\}$$

and

$$T = \{\, (x, y, z) \in \mathbb{R}^3 : (z + 1)^2/4 = x^2 + y^2, z \ge -1 \,\}.$$

i) Find the z-coordinates of the points of intersection of S and T and sketch the projection into the xy-plane of the curves of intersection.

ii) Sketch the section of S and T by the plane $x = 0$ on the same diagram.

iii) For what values of a do the surfaces $x^2 + y^2 + (z - 1)^2 = 1$ and $a(z + 1)^2 = x^2 + y^2$ $(z \ge -1)$ not intersect?

21: Find the projection into the xy-plane of the curve of intersection of the surfaces $2z = x^2 - y^2 + 2x$ and $3z = 4x^2 + y^2 - 2x$ and express its equation in polar co-ordinates.

[M] **22**: Sketch the surfaces given parametrically as follows and use `plot3d` in Maple; or `splot` with `set parametric` in Gnuplot, to check your answers.

i) $(x, y, z) = (\cos u \sin v, \sin u \sin v, \cos v)$, $0 \le u \le 2\pi$, $0 \le v \le \pi/2$.

ii) $(x, y, z) = (\cos u \cosh v,\ \sin u \cosh v,\ \sinh v)$, $0 \le u \le 2\pi$, $v \in \mathbb{R}$.

iii)*
$$\left.\begin{aligned} x &= \big(2 + v\sin(u/2)\big)\cos u \\ y &= \big(2 + v\sin(u/2)\big)\sin u \\ z &= v\cos(u/2) \end{aligned}\right\}$$
for $0 \le u \le 2\pi$, $-1 \le v \le 1$.

[Hints: Let (r, θ) be polar coordinates in the xy-plane, so that points in $\mathbb{R}^3$ are described by cylindrical coordinates (r, θ, z). Show that the intersection of the surface with the half-plane $\theta = u_0$ is the curve $(r, \theta, z) = \big(2 + v\sin(u_0/2), u_0, v\cos(u_0/2)\big)$, $-1 \le v \le 1$. Verify that this is a line segment and work out how its position changes as u_0 varies from 0 to 2π.]

Answers to problems

A1: $(1, 1)$ and $(4, 4)$.
A3: webnotes[1] **A4**: i) $1/5\left(-3/\sqrt{2}, 3/\sqrt{2}, 4\right)$, $\left(3/\sqrt{2}, 3/\sqrt{2}, \pi\right) + \lambda\left(-3/\sqrt{2}, 3/\sqrt{2}, 4\right)$. ii) $(1, 2, 3)/\sqrt{14}$, $(1, 1, 1) + \lambda(1, 2, 3)$. **A5**: i) ii) $(0, 0)$, $\pi/2$; $(2, 0)$, $\cos^{-1}(.8)$. iii) $(3/4, -1/4)$, $(3/4, 1/4)$ iv) $(-1/4, 3/4)$, $(-1/4, -3/4)$. **A7**: webnotes[2] **A9**: i) $(3, -8)$, $(0, 0)$, $(3, 8)$ **A10**: i) Use $v(t)^2 = \|\boldsymbol{v}(t)\|^2 = \boldsymbol{v}(t)\cdot\boldsymbol{v}(t)$. ii) $\boldsymbol{F}(t)\cdot\boldsymbol{v}(t) = m\boldsymbol{v}'(t)\cdot\boldsymbol{v}(t) = (\lambda\boldsymbol{r}(t)\times\boldsymbol{v}(t))\cdot\boldsymbol{v}(t) = 0$. **A17**: i) paraboloid ii) cone iii) sphere iv) cylinder iv) hyperboloid v) hyperboloid of 2 sheets. **A18**: (ii) circle $z = a$, $x^2 + y^2 = a^2$, (iii) disc $x^2 + y^2 \le a^2$ **A19**: $\boldsymbol{r}(t) = (\cos t, \sin t, \pm\sin t)$; ellipse; circle $\boldsymbol{r}(t) = (\cos t, \sin t, 0)$. **A20**: iii) $a > 1/3$ **A21**: $r = 2\cos\theta$, $0 \le \theta < 2\pi$

1.3 Metrics

23: i) Two metrics ρ and δ are said to be *topologically equivalent* if and only if every ρ-ball contains a δ-ball and every δ-ball contains a ρ-ball.

ii) Two metrics ρ and δ are said to be *equivalent* if and only if there are constants
$$c_1, c_2 > 0$$
such that
$$c_1\,\rho(\mathbf{x}, \mathbf{y}) \le \delta(\mathbf{x}, \mathbf{y}) \le c_2\rho(\mathbf{x}, \mathbf{y}).$$

Show that equivalent metrics (as defined in the notes) are topologically equivalent. [Note that the converse is not true.]

[H] **24**: Consider the two metrics $d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||$ and $\delta(\mathbf{x}, \mathbf{y}) = d(\mathbf{x}, \mathbf{y})/(1 + d(\mathbf{x}, \mathbf{y}))$ (you may assume they are metrics).

i) Show that d and δ are not equivalent.

ii) Show that for any $r > 0$ there is $R > 0$ such that
$$B_d(\mathbf{x}, r) \subseteq B_\delta(\mathbf{x}, R).$$

iii) Show that for any $0 < R < 1$ there is $r > 0$ such that
$$B_\delta(\mathbf{x}, R) \subseteq B_d(\mathbf{x}, r).$$

iv) Are the metrics δ and d topologically equivalent (topological equivalence is defined in the previous question).

[H] **25**: i) Let d be the usual distance function on $\mathbb{R}^n$ (i.e. d_2) and let d_∞ be defined by $d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \max\{\,|x_1 - y_1|, \ldots, |x_n - y_n|\,\}$. Show that d and d_∞ are equivalent metrics on $\mathbb{R}^n$ by showing that
$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) \le d(\boldsymbol{x}, \boldsymbol{y}) \le \sqrt{n}\,d_\infty(\boldsymbol{x}, \boldsymbol{y})$$
for all $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^n$.

ii) Suppose that $\boldsymbol{f} : \mathbb{R}^m \to \mathbb{R}^n$ has component functions $f_i : \mathbb{R}^m \to \mathbb{R}$ for $i = 1, \ldots, n$ and that $f_i(\boldsymbol{x}) \to b_i$ as $\boldsymbol{x} \to \boldsymbol{a}$ for $i = 1, \ldots, n$. Prove, from the definition of limits, that $\boldsymbol{f}(\boldsymbol{x}) \to \boldsymbol{b}$ as $\boldsymbol{x} \to \boldsymbol{a}$. [Hint: Use d_∞ instead of d.]

iii) Show that if d_p is defined for positive integers p by
$$d_p(\boldsymbol{x}, \boldsymbol{y}) = \big(|x_1 - y_1|^p + \ldots + |x_n - y_n|^p\big)^{1/p}$$
then $d_p(\boldsymbol{x}, \boldsymbol{y}) \to d_\infty(\boldsymbol{x}, \boldsymbol{y})$ as $p \to \infty$. [This is a slight generalisation of a problem in the MATH1231/1241 problem booklet which asks you to find the limit as $n \to \infty$ of $(a^n + b^n)^{1/n}$ when $a \ge b > 0$.]

i) webnotes[3] ii) webnotes[4]

[1] http://web.maths.unsw.edu.au/~potapov/2111_2015/Week-1-Lecture-1.html#g_t_0060_0060ABC_0027_0027-curve-in-gnuplot-_0028video_0029
[2] http://web.maths.unsw.edu.au/~potapov/2111_2015/Algebraic-properties-of-derivative-of-curve.html
[3] http://web.maths.unsw.edu.au/~potapov/2111_2015/Metrics-and-equivalence-theorem.html
[4] http://web.maths.unsw.edu.au/~potapov/2111_2015/Further-properties-of-limits-of-vector-map.html

2 Open and Closed subsets; Limits

[M] – Maple/Gnuplot; [A] – additional/optional problems; [H] – harder problems.

2.1 Open and Closed subsets of $\mathbb{R}^n$

In this subsection, you are only allowed to use definitions of open and closed sets and definition of the boundary of a set.

26: Show that

1) $[a, b]$ is closed, $a, b \in \mathbb{R}$;
2) (a, b) is open, $a, b \in \mathbb{R}$;
3) $\emptyset$ is open and close;
4) $\mathbb{R}$ is open and close;
5) $[a, b)$ is neither closed nor open, $a, b \in \mathbb{R}$;
6) $\mathbb{Q}$ is neither closed nor open;
7) $\Big\{k^{-1} : \; k \in \mathbb{Z}, k \neq 0\Big\}$ is neither open nor closed;
8) The open ball $B(\mathbf{x}, \epsilon)$ is open.

27: Determine whether or not the set

$$\Big\{(m^{-1}, n^{-1}) : \quad m, n \in \mathbb{Z}, \quad m, n > 0\Big\}$$

is closed.

28: Let

$$\Omega = \Big\{(x, y) \in \mathbb{R}^2 : x + y \neq 0\Big\}.$$

Show that Ω is an open subset of $\mathbb{R}^2$.

[A] **29**:

i) If Ω_1 and Ω_2 are open sets in $\mathbb{R}^n$, show that $\Omega_1 \cap \Omega_2$ and $\Omega_1 \cup \Omega_2$ are open.

ii) If Ω_1 and Ω_2 are closed sets in $\mathbb{R}^n$, show that $\Omega_1 \cap \Omega_2$ and $\Omega_1 \cup \Omega_2$ are closed.

30: Show that every point $(0, a)$ with $|a| \leq 1$ is the boundary point of the set

$$S = \Big\{(x, y) \in \mathbb{R}^2 : x > 0, \quad y = \sin(1/x)\Big\}.$$

2.2 Limits

31: Use definition of the limit to show that

i) $\displaystyle\lim_{x\to 2} \frac{x+1}{x+2} = \frac{3}{4}$.

ii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 + x^2 + y^2 + y^4}{x^2 + y^2} = 1$.

32: Show that the following limits do not exist

i) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$;

ii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$.

33: For the limits below give two proofs: one using *pinching principle* and one using the definition of the limit directly

i) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2 + y^2}$;

ii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2 + x^2 y^2}{x^2 + y^2} = 1$.

34: Let

$$f(x, y) = \frac{x - y}{x + y}.$$

Show that

$$\lim_{x\to 0} \left[\lim_{y\to 0} f(x, y)\right] = 1 \quad \text{and} \quad \lim_{y\to 0} \left[\lim_{x\to 0} f(x, y)\right] = -1.$$

Show also that

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

does not exist.

35: Let

$$f(x, y) = \frac{x^2 y^2}{x^2 y^2 + (x - y)^2}.$$

Show that

$$\lim_{x\to 0} \left[\lim_{y\to 0} f(x, y)\right] = \lim_{y\to 0} \left[\lim_{x\to 0} f(x, y)\right] = 0.$$

Show also that

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

does not exist.

36: Let

$$f(x,y) = (x+y)\,\sin\frac{1}{x}\,\sin\frac{1}{y}, \quad x\neq 0,\ y\neq 0$$

and

$$f(x,y) = 0, \quad x = 0 \text{ or } y = 0.$$

Show that neither

$$\lim_{y\to 0} f(x,y),\ x\neq 0 \quad\text{nor}\quad \lim_{x\to 0} f(x,y),\ y\neq 0$$

exist. Also, use *pinching principle* to show that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

37: Use *pinching principle* to show that

$$\lim_{(x,y)\to(0,0)} \frac{xy(x+y)}{x^2-xy+y^2} = 0.$$

Hint: Prove first that

$$\left|\frac{xy}{x^2-yx+y^2}\right| \le 1, \quad \forall (x,y)\neq 0.$$

38: Use *pinching principle* to show that

$$\lim_{(x,y)\to(0,0)} \frac{xy(x+y)}{x^2+y^2} = 0.$$

Hint: Prove first that

$$\frac{|xy|}{x^2+y^2} \le \frac{1}{2}.$$

2.3 Limits and Taylor expansions

In the following questions you are allowed to use the known Taylor expansions below. In the expansions below the function $\epsilon(x)$ different from one expansion to another and is such that

$$\epsilon(x): \quad \lim_{x\to 0}\epsilon(x) = 0$$

Taylor expansions

$$e^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!} + x^n\epsilon(x) = \sum_{k=0}^{n}\frac{x^k}{k!} + x^n\epsilon(x)$$

$$\sin x = x - \frac{x^3}{3!} + \ldots + (-1)^k\frac{x^{2n+1}}{(2n+1)!} + x^{2n+1}\epsilon(x) = \sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{(2k+1)!} + x^{2n+1}\epsilon(x)$$

$$\sinh x = x + \frac{x^3}{3!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + x^{2n+1}\epsilon(x) = \sum_{k=0}^{n}\frac{x^{2k+1}}{(2k+1)!} + x^{2n+1}\epsilon(x)$$

$$\cos x = 1 - \frac{x^2}{2} + \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + x^{2n}\epsilon(x) = \sum_{k=0}^{n}(-1)^k\frac{x^{2k}}{(2k)!} + x^{2n}\epsilon(x)$$

$$\cosh x = 1 + \frac{x^2}{2} + \ldots + \frac{x^{2n}}{(2n)!} + x^{2n}\epsilon(x) = \sum_{k=0}^{n}\frac{x^{2k}}{(2k)!} + x^{2n}\epsilon(x)$$

$$\ln(1+x) = x - \frac{x^2}{2} - \ldots + (-1)^{n-1}\frac{x^n}{n} + x^n\epsilon(x) = \sum_{k=1}^{n}(-1)^{k+1}\frac{x^k}{k} + x^n\epsilon(x)$$

$$(1+x)^\alpha = 1 + \alpha x + \ldots + \binom{\alpha}{n} x^n + x^n \epsilon(x) = \sum_{k=0}^{n} \binom{\alpha}{k} x^k + x^n \epsilon(x)$$

$$\binom{\alpha}{k} = \prod_{s=1}^{k} \frac{\alpha - s + 1}{s} = \frac{\alpha \times (\alpha - 1) \times \ldots \times (\alpha - k + 1)}{k!}$$

39: Prove that

$$\lim_{(x,y)\to(0,a)} \frac{\sin(xy)}{x} = a.$$

40: Find

$$\lim_{(x,y)\to(a,0)} \frac{1-\cos(xy)}{y^2}.$$

41: Find

$$\lim_{(x,y)\to(a,0)} \frac{\ln(1+xy)}{y}.$$

42: Find

$$\lim_{(x,y)\to(0,a)} \frac{(x+y)^{2/3} - y^{2/3}}{x}.$$

Answers to problems

A26: Direct argument for part (8) is given in these webnotes[5]. **A34**: See these webnotes[6] for an idea how to show that a limit does not exist **A35**: See answer to Problem 34 **A36**: See these webnotes[7] with an example of argument showing that limit exists **A37**: See the answer to Problem 36 **A38**: See the answer to Problem 36 **A38**: See the answer to Problem 36

2.4 Limits and metrics

43: For $k \geq 1$ let

$$\boldsymbol{x}_k = \left(\frac{3k-1}{k+3}, \frac{2k+2}{k+3} \right),$$

and let

$$\boldsymbol{x} = (3,2).$$

i) Calculate $d_2(\boldsymbol{x}_k, \boldsymbol{x})$ and $d_\infty(\boldsymbol{x}_k, \boldsymbol{x})$.

ii) Fix $\epsilon > 0$. Find a K such that for all $k \geq K$, $d_2(\boldsymbol{x}_k, \boldsymbol{x}) < \epsilon$. Do the same for d_∞ in place of d_2.

iii) Does $\boldsymbol{x}_k \to \boldsymbol{x}$ as $k \to \infty$?

2.5 Limits and continuity

[H] **44**: Show (by ϵ, δ argument) that the following function is continuous at $(0,0)$:

$$f(x,y) = \begin{cases} \dfrac{x^4+y^4}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

Hint: These webnotes[8] have a similar argument.

45: If

$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}, \quad (x,y) \neq (0,0),$$

how must $f(0,0)$ be defined so as to make f continuous at the origin? *Hint:*

$$\left| \frac{\sin a}{a} - 1 \right| \leq |a|^2, \quad a \in \mathbb{R}.$$

You do not need to prove this inequality.

46: Let

$$f(x,y) = \frac{xy}{x^2+y^2}, \quad (x,y) \neq (0,0).$$

By considering appropriate curves which approach $(0,0)$ (or otherwise), show that there is no value of $f(0,0)$ which makes the function f continuous at the origin.

[5] http://web.maths.unsw.edu.au/~potapov/2111_2015/A-ball-is-open-subset.html
[6] http://web.maths.unsw.edu.au/~potapov/2111_2015/Limit-by-sequences-_002d_002d-Example.html
[7] http://web.maths.unsw.edu.au/~potapov/2111_2015/Example-of-limit-of-vector-map.html
[8] http://web.maths.unsw.edu.au/~potapov/2111_2015/Example-of-limit-of-vector-map.html

See these webnotes[9]

47: For each of the following functions $f : \mathbb{R} \to \mathbb{R}$, find an open set U in $\mathbb{R}$ such that $f^{-1}(U)$ is not open.

i) $f(x) = 1$ for $x \geq 0$ and 0 otherwise.

ii) $f(x) = 1/x$ for $x > 0$ and 0 otherwise.

[H] **48**: Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is such that the sets

$$\Big\{\boldsymbol{x} \in \mathbb{R}^n : f(x) > d\Big\}$$

and

$$\Big\{\boldsymbol{x} \in \mathbb{R}^n : f(x) < d\Big\}$$

are open for every $d \in \mathbb{R}$. Prove that f is continuous.

3 Continuous maps and classes of subsets

[M] – Maple/Gnuplot; [A] – additional/optional problems; [H] – harder problems.

3.1 Open and closed subsets

You are allowed to use Problem 50 to prove your answer to other problems in this subsection. Also, you are allowed to use the fact that an interval (and a disjoint union of intervals) in $\mathbb{R}^1$ is open (closed, respectively) if and only if it does not contain its end points (contains all of its end points, respectively).

49: Prove that, for any subsets $U, V \in \mathbb{R}^m$ and every map

$$\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^m,$$

i) $f^{-1}(U^c) = \big(f^{-1}(U)\big)^c$;

ii) $f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V)$;

iii) $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$.

[A] **50**: Let f be a function from $\mathbb{R}^m$ to $\mathbb{R}^n$.

i) Prove that f is continuous on $\mathbb{R}^m$ if and only if $f^{-1}(U)$ is open in $\mathbb{R}^m$ for all open sets U in $\mathbb{R}^n$.

ii) Use the part i) of Question 49 and deduce that f is continuous on $\mathbb{R}^m$ if and only if $f^{-1}(U)$ is closed in $\mathbb{R}^m$ for all closed sets U in $\mathbb{R}^n$.

51: Decide whether each of the following subsets is open or closed. Prove your answer.

i)

$$\Big\{(x, y) \in \mathbb{R}^2 : \quad x^2 - y^2 < 1\Big\}.$$

ii)

$$\Big\{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 1\Big\}.$$

iii)

$$\Big\{(x, y) \in \mathbb{R}^2 : \quad x^2 - y^2 \geq 1\Big\}.$$

iv)

$$\Big\{(x, y, z) \in \mathbb{R}^3 : \quad \sqrt{x^2 + y^2} \leq z \leq 1\Big\}.$$

52: For each set Ω below show that Ω is neither open nor closed. Do so, by finding $f^{-1}(\Omega)$ for the function f provided. Find suitable function f if no function is given.

i) $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x, 0)$,

$$\Big\{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \leq 1\Big\}.$$

ii) $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x, 1)$

$$\Big\{(x, y) \in \mathbb{R}^2 : \quad 0 \leq x^2 - y^2 < 1\Big\}.$$

iii)

$$\Big\{(x, y, z) \in \mathbb{R}^3 : \quad \sqrt{x^2 + y^2} \leq z < 1\Big\}.$$

53: Determine the interior and boundary of the set

$$\Big\{(x, y) \in \mathbb{R}^2 : \quad 0 < x^2 + y^2 < 1\Big\}.$$

[H] **54**: Show that a subset S of $\mathbb{R}^n$ is closed if and only if it contains all its boundary points.

55: For the following sets S determine:

i) the boundary of S

ii) the interior of S

iii) whether S is open, closed or neither.

[9] http://web.maths.unsw.edu.au/~potapov/2111_2015/Limit-by-sequences-_002d_002d-Example.html

i) $S = \Big\{(x,y) \in \mathbb{R}^2 : \quad x^2 - y^2 < 1\Big\}$.

ii) $S = \Big\{(x,y) \in \mathbb{R}^2 : \quad x^2 - y^2 \geq 1\Big\}$.

iii) $S = \Big\{(x,y,z) \in \mathbb{R}^3 : \quad \sqrt{x^2+y^2} \leq z \leq 1\Big\}$.

iv)*

$$S = \Big\{(x,y) \in \mathbb{R}^2 : \Big(0 < |x| \leq 1,\ y = \sin\frac{1}{x}\Big) \text{ or } \Big(x = 0,\ -1 \leq y \leq 1\Big)\Big\}.$$

3.2 Bounded and path-connected subsets

56: Decide whether the subsets of Question 51 are bounded or unbounded. Prove your answer.

57: Decide whether the following subsets are path-connected or not path-connected. Prove your answer. You allowed to use both the definition of path-connected subsets, IVT, Problem 58 and polar coordinates:

$$x = r\cos t,\\ y = r\sin t$$

i)

$$\Big\{(x,y) \in \mathbb{R}^2 : \quad x \in [a,b],\ y \in [c,d]\Big\}.$$

ii)

$$\Big\{(x,y) \in \mathbb{R}^2 : \frac{1}{2} < x^2 + y^2 < 1\Big\}.$$

iii)

$$\Big\{(x,y) \in \mathbb{R}^2 : \frac{1}{2} < x^2 + y^2 < 1 \text{ and } xy > 0\Big\}.$$

3.3 Continuous maps and subsets

[A] **58**: Let $\mathbf{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be continuous. Prove that

i) $K \subset \Omega$ and K is compact $\Rightarrow \mathbf{f}(K)$ is compact.

ii) $B \subset \Omega$ and B is path connected $\Rightarrow \mathbf{f}(B)$ is path connected.

59: Consider

$$S_1 = \Big\{(x,y) \in \mathbb{R}^2 \ : \ \frac{1}{4} \leq x^2 + y^2 \leq 1,\ x,y \geq 0\Big\}$$
$$S_2 = \Big\{(x,y) \in \mathbb{R}^2 \ : \ \frac{1}{4} \leq x^2 + y^2 < 1,\ x,y \geq 0\Big\}$$
$$Q = \Big\{(x,y) \in \mathbb{R}^2 \ : \ \frac{1}{4} \leq x^2 + y^2,\ x,y \geq 0\Big\}$$

Is there a continuous function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ such that

i) $\mathbf{f}(S_1) = S_2$?

ii) $\mathbf{f}(S_2) = S_1$?

iii) $\mathbf{f}(Q) = S_2$?

iv) $\mathbf{f}(Q) = S_1$?

v) $\mathbf{f}(S_2) = Q$?

vi) $\mathbf{f}(S_1) = Q$?

Hint: Consider using polar and inverse polar map to reduce the problem to one-dimension and map the radius component only.

[H] **60**: In the context of Problem 59, consider the subsets

$$S_1 = \Big\{(x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq 1\Big\}$$
$$S_2 = \Big\{(x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 < 1\Big\}$$
$$Q = \mathbb{R}^2$$

Is it possible to use the polar and inverse polar maps in this case?

[H] **61**: Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x,y) = (x^2 - y^2, x + y + 1).$$

Find the images of the following sets under f.

i) $\{(x,y) \ : \ x \geq 0,\ y \geq 0\}$.

ii) $\{(x,y) \ : \ 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$.

iii) $\{(x,y) \ : \ y \geq -x\}$.

Hint: Map boundary first. In case of part (iii), find the image of the straight lines $x + y = a$ with $a > 0$.

Answers to problems

A50: Part i) proved in these webnotes[10] **A52**: Argument for (i) is explained in these webnotes[11]; Also, if you are interested in the direct, based on definition only, argument for (i), look at these webnotes[12];

A53: These webnotes[13] have possible solution **A55**: i) open. ii) closed. iii) closed. iv) closed.

3.4 Properties of continuous functions

[H] **62**: (Algebra of continuous functions)

i) For two functions continuous on Ω

$$f : \Omega \subset \mathbb{R}^n \to \mathbb{R} \quad \text{and} \quad g : \Omega \subset \mathbb{R}^n \to \mathbb{R},$$

prove that $f + g$, fg and f/g are continuous. [The domain of f/g must exclude points where $g(\mathbf{x}) = 0$.]

ii) For two continuous functions

$$f : \mathbb{R}^m \to \mathbb{R} \quad \text{and} \quad g : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$$

prove that $f \circ g$ is continuous where it makes sense.

These webnotes[14] present a similar argument for limits

[H] **63**: Suppose that $\mathbf{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{x}_0 \in \Omega$. Prove that the following are equivalent

i) $\mathbf{f}$ is continuous at $\mathbf{x}_0 \in \Omega$.

ii) $\forall \epsilon > 0$, $\exists \delta > 0$ such that for $\mathbf{x} \in \Omega$,

$$d(\mathbf{x}, \mathbf{x}_0) < \delta \;\Rightarrow\; d(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{x}_0)) < \epsilon,$$

i.e.

$$\mathbf{x} \in B(\mathbf{x}_0, \delta) \Rightarrow \mathbf{f}(\mathbf{x}) \in B(\mathbf{f}(\mathbf{x}_0), \epsilon).$$

iii) $\forall$ sequences $\{\mathbf{x}_k\}$ in Ω with limit $\mathbf{x}_0$, $\{\mathbf{f}(\mathbf{x}_k)\}$ converges to $\mathbf{f}(\mathbf{x}_0)$.

iv) $\mathbf{f}(\mathbf{x}_0)$ is an interior point of $\mathbf{f}(\Omega)$ $\Rightarrow$ $\mathbf{x}_0$ is an interior point of Ω.

ii) see definition of limit in these webnotes[15] iii) See connection between limit and sequences in these webnotes[16] iv) Use Question 50

See these webnotes[17] for the proof

64: For each of the sets in question 55 determine whether they are bounded, compact, and/or path connected.

i) connected. ii) none. iii) bounded, compact, path connected. iv) bounded, compact, not path connected!

65: Let

$$\begin{aligned}
X_1 &= \{(x, y) \,:\, 0 \le x \le 1,\ 0 \le y \le 1\} \\
X_2 &= \{(x, y) \,:\, x^2 + y^2 < 1\} \\
X_3 &= \{(x, y) \,:\, x^2 < y^2 < 1\} \\
X_4 &= \{(x, y) \,:\, x \ge 0,\ y \ge 0\}
\end{aligned}$$

For each pair $i \neq j$, decide whether there is any **continuous** function f_{ij} which maps X_i **onto** X_j. (A formula for f_{ij} is not required if you can describe the action of the function.) These webnotes[18] show a similar example with detailed argument.

X_1 can't be mapped onto any of the other sets; X_2 can be mapped onto X_1, X_4; X_3 onto all of the others; X_4 onto X_1, X_2.

[10] http://web.maths.unsw.edu.au/~potapov/2111_2015/Continuity-via-preimage.html
[11] http://web.maths.unsw.edu.au/~potapov/2111_2015/Continuity-via-preimage-_002d_002d-Examples.html
[12] http://web.maths.unsw.edu.au/~potapov/2111_2015/Non_002dtrivial-example-of-open-subset.html
[13] http://web.maths.unsw.edu.au/~potapov/2111_2015/Open-and-closed-subsets-under-set-operations.html
[14] http://web.maths.unsw.edu.au/~potapov/2111_2015/Week-1-Lecture-1.html#index-Theorem_002c-algebraic-properties-of-continuous-cur
[15] http://web.maths.unsw.edu.au/~potapov/2111_2015/Limit-of-vector-map.html
[16] http://web.maths.unsw.edu.au/~potapov/2111_2015/Limit-of-vector-map-via-sequences.html
[17] http://web.maths.unsw.edu.au/~potapov/2111_2015/Image-of-path_002dconnected_002fcompact-subsets.html
[18] http://web.maths.unsw.edu.au/~potapov/2111_2015/Image-of-path_002dconnected_002fcompact-subsets.html#Examples-of-usage-of-the-theorem

4 Differentiation

4.1 Partial derivatives and Jacobians

66: Find all first and second order partial derivatives for the function

$$z = x^5 + y^5 - 3x^3y^3.$$

67: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{(x^2+y^2)} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

i) Calculate

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}$$

first for $(x,y) \neq (0,0)$ (you can use Maple if you like) and then for $(x,y) = (0,0)$.

ii) Show that

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0).$$

Discuss!

68: Let

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{otherwise} \end{cases}$$

Does the derivative

$$\frac{\partial^2 f}{\partial x \partial y}(0,0)$$

exist?

69: Find $\dfrac{\partial f}{\partial y}(1,y)$ for the function

$$f(x,y) = x^{x^{x^y}} + (\ln x) \times \tan^{-1}\Big[\tan^{-1}\Big(\sin\big[\cos(xy) - \ln(x+y)\big]\Big)\Big].$$

70: Find a general formula for the Jacobian matrix of the function $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\boldsymbol{f}(x,y,z) = \begin{bmatrix} xy\sin z \\ xy\cos z \\ x^2+y^2+z^2 \end{bmatrix}$$

and find its value at the point $(2,1,0)$.

71: Verify that the equation

$$J(\boldsymbol{f} \cdot \boldsymbol{g}) = \boldsymbol{g}^T \times J\boldsymbol{f} + \boldsymbol{f}^T \times J\boldsymbol{g}$$

holds in the case where

$$\boldsymbol{f}, \boldsymbol{g} : \mathbb{R}^n \mapsto \mathbb{R}^n.$$

.

4.2 Definition of differentiability, properties

72: If $f : \mathbb{R}^n \mapsto \mathbb{R}$ and $\boldsymbol{a} \in \mathbb{R}^n$, show that there cannot be two different linear functions

$$\ell : \mathbb{R}^n \to \mathbb{R}$$

satisfying

$$\frac{f(\boldsymbol{a}+\boldsymbol{x}) - f(\boldsymbol{a}) - \ell(\boldsymbol{x})}{\|\boldsymbol{x}\|} \to 0 \quad \text{as} \quad \boldsymbol{x} \to \boldsymbol{0}.$$

73: Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x,y,z) = xy + yz + xz.$$

Show, using the definition of differentiability (see these webnotes[19]), that f is differentiable at the point $(1,1,1)$.

74: Let

$$f(x,y) = \sqrt[3]{xy}, \quad x,y \in \mathbb{R}.$$

Find

$$f_x(0,0) \quad \text{and} \quad f_y(0,0).$$

Is this function differentiable at $(0,0)$?

75: Let

$$f(x,y) = \sqrt[3]{x^3+y^3}, \quad x,y \in \mathbb{R}.$$

Find

$$f_x(0,0) \quad \text{and} \quad f_y(0,0).$$

Is this function differentiable at $(0,0)$?

[19] http://web.maths.unsw.edu.au/~potapov/2111_2015/Differentiability-of-vector-map.html

76: Let

$$f(x,y) = \begin{cases} \dfrac{\sin\left(x^2+y^2\right)}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 1, & \text{otherwise} \end{cases}$$

Find

$$f_x(0,0) \quad \text{and} \quad f_y(0,0)$$

and show that this function is differentiable at $(0,0)$.

77: For a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ and a point $(x_0, y_0) \in \mathbb{R}^2$ define

$$\Delta f = f(x,y) - f(x_0,y_0)$$

and define

$$\Delta x = x - x_0 \quad \text{and} \quad \Delta y = y - y_0.$$

Define also

$$df = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y.$$

In many problems, the value of Δf is approximated by the value of df. This explained by the fact that

$$\lim_{(x,y)\to(x_0,y_0)} |\Delta f - df| = 0.$$

Prove the limit above.

[H] **78**: Let

$$f(x,y) = \begin{cases} \exp\left(-\frac{1}{x^2+y^2}\right), & \text{if } (x,y) \neq (0,0) \\ 0, & \text{otherwise} \end{cases}$$

Find

$$f_x(0,0) \quad \text{and} \quad f_y(0,0)$$

and show that this function is differentiable at $(0,0)$. *Hint:* Use the fact that exponential function decays faster than any power function, i.e., for every integer $n \in \mathbb{N}$, there is a constant $B_n > 0$ such that

$$0 < \exp\left(-a^{-1}\right) \leq B_n a^n, \quad \forall a \in (0,1).$$

You do not need to prove this inequality.

4.3 Best affine approximations

79: What is the best affine approximation to the function $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\boldsymbol{f}(x,y) = \begin{bmatrix} e^{xy^2} \\ x^2 - 3x + y^2 \end{bmatrix}$$

at the point $(1, -1)$.

80: When two resistances r_1 and r_2 are connected in parallel, the total resistance R (measured in ohms) is given by:

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}.$$

i) Show that $\dfrac{\partial R}{\partial r_1} = \dfrac{R^2}{r_1^2}$.

ii) Use *the best affine approximation* of function $R(r_1, r_2)$, to estimate the maximum possible error in the calculated value of R if the measured values of r_1 and r_2 are $r_1 = 6 \pm 0.1$ ohms and $r_2 = 9 \pm 0.03$ ohms

81: The specific gravity δ of a solid heavier than water is given by

$$\delta = \frac{W}{W - W_1}$$

where W and W_1 are its weight in air and water respectively. W and W_1 are observed to by 17.2 and 9.7 gm. Use *the best affine approximation* of function $\delta(W, W_1)$ to estimate the maximum possible error in the calculated value of δ due to an error of 0.05 gm in each observation.

4.4 Chain Rule, First order

82: Let $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ and $\boldsymbol{g} : \mathbb{R}^m \to \mathbb{R}^p$ and $\boldsymbol{h} = \boldsymbol{g} \circ \boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^p$ and let $\mathbf{a} \in \mathbb{R}^n$. For each of the examples below find the left hand side and the right hand side of the chain rule identity:

$$J_{\mathbf{a}}\boldsymbol{h} = J_{\boldsymbol{f}(\mathbf{a})}\boldsymbol{g} \times J_{\mathbf{a}}\boldsymbol{f}.$$

i)

$$\boldsymbol{f}(x,y,z) = \begin{bmatrix} x^2 - y^2 \\ 2xy \\ z \end{bmatrix},$$

$$\boldsymbol{g}(u,v,w) = \begin{bmatrix} u + w^2 \\ u/w \end{bmatrix},$$

$$\boldsymbol{a} = (2,1,2).$$

ii)

$$\boldsymbol{f}(x,y) = \begin{bmatrix} x^2 + y \\ x - 2y^2 \end{bmatrix},$$

$$\boldsymbol{g}(u,v) = \begin{bmatrix} 2u + v \\ \sin u \\ u + 2v^2 \end{bmatrix},$$

$$\boldsymbol{a} = (1,1);$$

iii)

$$g(x,y) = \sqrt{x^2 + y^2},$$

$$\boldsymbol{f}(s,t) = \begin{bmatrix} e^{st} \\ 1 + s^2 \cos t \end{bmatrix},$$

$$\boldsymbol{a} = (1,0).$$

iv)

$$g(x,y) = e^{xy^2},$$

$$\boldsymbol{f}(t) = \begin{bmatrix} t\,\cos t \\ t\,\sin t \end{bmatrix},$$

$$a = \frac{\pi}{2}.$$

83: A function $f(x,y)$ is said to be *homogeneous* of degree m if $f(tx,ty) = t^m f(x,y)$ for every real number $t > 0$. Euler's theorem states that if f is homogeneous of degree m and if all its partial derivatives of first order exist and continuous then

$$x\,\frac{\partial f}{\partial x} + y\,\frac{\partial f}{\partial y} = mf(x,y).$$

i) Verify Euler's theorem for

$$f(x,y) = Ax^2 + Bxy + Cy^2$$

and for

$$g(x,y) = \tan^{-1}\frac{y}{x}, \quad x \neq 0.$$

ii) Prove Euler's theorem.

iii) Generalise the theorem and prove your generalisation.

84: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and

$$z = xy + f\left(\frac{y}{x}\right), \quad (x,y) \in \mathbb{R}^2, \quad x \neq 0.$$

Show that z satisfies the partial differential equation

$$x\,\frac{\partial z}{\partial x} + y\,\frac{\partial z}{\partial y} = 2xy.$$

85: Find $\partial w/\partial t$ if

$$w = f(x,y,z)$$

and

$$x = g(s,t), \quad y = h(s,t) \quad \text{and} \quad z = k(s,t).$$

4.5 Directional derivatives

86: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

Show that for all unit vectors $\boldsymbol{u}$ the directional derivative of f at the origin in the direction $\boldsymbol{u}$ does exist, but f is discontinuous at $(0,0)$. Show that there is no plane which contains all the lines which are tangent to the surface $z = f(x,y)$ at $(0,0,0)$.

87: For each of the following scalar fields

a) find ∇f

b) graph some level curves $f(x,y) =$ constant,

c) indicate ∇f at some points by arrows on these curves.

i) $f(x,y) = xy$

ii) $f(x,y) = x^2 + y^2$

iii) $f(x,y) = \dfrac{y}{x^2}$.

88: Let $\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$ and $r = \|\boldsymbol{r}\|$.

i) Prove that $\nabla r = \dfrac{\boldsymbol{r}}{r}$ and $\nabla\left(\dfrac{1}{r}\right) = \dfrac{-\boldsymbol{r}}{r^3}$.

ii) Calculate $\nabla(\cos r)$, $\nabla\left(\dfrac{\log r}{r}\right)$.

iii) Prove that $\nabla r^n = nr^{n-2}\boldsymbol{r}$.

89: In each case find ∇f at the point P and use it to find the directional derivative of f at P in the direction of $\boldsymbol{v}$.

i) $f(x,y) = 13x^2 + 7xy + 2y$, $P = (-1,1)$, $\boldsymbol{v} = 5\boldsymbol{i} + 12\boldsymbol{j}$.

ii) $f(x,y,z) = x(x^2+y^2+z^2)$, $P = (1,2,-1)$, $\boldsymbol{v} = \boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$.

90: Suppose $f(x,y)$ is a differentiable function, which has, at the point $\boldsymbol{x}$, directional derivative $1/\sqrt{2}$ in the direction $(1,1)$ and directional derivative $1/5$ in the direction $(3,4)$. Find $\nabla f(\boldsymbol{x})$.

91: A bushwalker is climbing a mountain, of which the equation is $h(x,y) = 400-(x^2+4y^2)/10000$. Here x, y and h are measured in metres, the x-axis points East and the y-axis points North. The bushwalker is at a point P, 1600 metres West and 400 metres South of the peak.

i) What is the slope of the mountain at P in the direction of the peak?

ii) In which direction at P is the slope greatest?

92: The electrical potential V is given by $V(x,y,z) = x^2 - xy + xyz$.

i) Find the rate of change of the potential V at $(1,1,1)$ in the direction of the vector $\boldsymbol{v} = \boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}$.

ii) In which direction(s) does V change most rapidly at $(1,1,1)$?

iii) What is the maximum rate of change of V at $(1,1,1)$?

93: Skier is on a mountain described by the equation $h(x,y) = 2000 - x^4/10^8 - y^2/10^2$ at the point $(100,1)$. He skis down the mountain, always moving in the direction of steepest descent.

i) In what direction does he start moving?

ii) Describe the curve along which he skis. [You will need to solve a separable first order ODE.]

94: Use the definition of directional derivative to compute the directional derivative for f at the point P in the direction $\boldsymbol{u}$.

i) $f(x,y) = 2 - x^2/2 - y^2$, $P = (1, 1/\sqrt{2})$, $\boldsymbol{u} = (1/\sqrt{2}, -1/\sqrt{2})$;

ii) $f(x,y) = \sin(x^2-y^2)$, $P = (\pi,\pi)$, $\boldsymbol{u} = (1,0)$;

[H] **95**: Let
$$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{R}^3$$
be an *orthonormal system* and let
$$f : \mathbb{R}^3 \mapsto \mathbb{R}$$
be differentiable. Prove that
$$\left(\frac{\partial f}{\partial \mathbf{u}_1}\right)^2 + \left(\frac{\partial f}{\partial \mathbf{u}_2}\right)^2 + \left(\frac{\partial f}{\partial \mathbf{u}_3}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2.$$

[H] **96**: i) Let $\mathbf{u}$ be a unit vector. Prove that
$$\frac{\partial^2 f}{\partial \mathbf{u}^2} = \mathbf{u}^T H \mathbf{u},$$
where H is the Hessian of f.

ii) Let
$$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{R}^3$$
be an *orthonormal system* and let
$$f : \mathbb{R}^3 \mapsto \mathbb{R}$$
be twice differentiable. Prove that
$$\frac{\partial^2 f}{\partial \mathbf{u}_1^2} + \frac{\partial^2 f}{\partial \mathbf{u}_2^2} + \frac{\partial^2 f}{\partial \mathbf{u}_3^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

4.6 Chain rule, Second order

97: Find
$$\frac{\partial^2 z}{\partial r^2}, \quad \frac{\partial^2 z}{\partial s^2}, \quad \frac{\partial^2 z}{\partial r \partial s}$$
in terms of f_x, f_y and f_{xx}, f_{xy} and f_{yy}, if $z = f(x,y)$ and $x = 3r+s$ and $y = r-s$.

[H] **98**: Let
$$f = f(x,y), \quad x, y \in \mathbb{R}$$
and let
$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Find
$$\frac{\partial f}{\partial r} \text{ and } \frac{\partial f}{\partial \theta}.$$
in terms of
$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$
and vice versa.

[H] **99**: Let
$$f = f(x,y), \quad x, y \in \mathbb{R}$$
and let
$$x = r\cos\theta \text{ and } y = r\sin\theta.$$

i) Find
$$\frac{\partial^2 f}{\partial r^2}, \ \frac{\partial^2 f}{\partial r \partial \theta} \text{ and } \frac{\partial f}{\partial \theta}.$$
in terms of
$$\frac{\partial f}{\partial x}, \ \frac{\partial^2 f}{\partial x \partial y} \text{ and } \frac{\partial f}{\partial y}.$$

ii) Find the converse relation.

[H] **100**: Use computations of Question 99 to compute the *Laplacian* operator
$$\Delta f = f_{xx} + f_{yy}$$
in polar coordinates
$$x = r\cos\theta \text{ and } y = r\sin\theta.$$

101: Let $z = f(x,y)$ and $x = e^u \cos v$ and $y = e^u \sin v$ and f have continuous second-order partial derivatives.

i) Show that
$$\frac{\partial^2 z}{\partial u^2} = x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} + x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}.$$

ii) Show that
$$\frac{\partial^2 z}{\partial u \partial v} = (x^2 - y^2)\frac{\partial^2 f}{\partial x \partial y} + xy\left(\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x^2}\right) - y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}.$$

102: If $g(t) = f(x,y,t)$ where $x = \cos t$ and $y = \sin t$, express the derivative of g in terms of the partial derivatives of f.

103: If $g(u,v) = f(x,u,v)$ where x is a function of u and v, express $\dfrac{\partial g}{\partial u}$ in terms of the partial derivatives of f and x.

104: Given that
$$u = f(x+ct) + g(x-ct),$$
where f, g are twice-differentiable functions of one variable and c is a constant, show that u satisfies the wave equation
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

105: A function f of two variables is called *harmonic* if it satisfies Laplace's equation
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$
Show that if $f(x,y)$ is harmonic then
$$f(x^2 - y^2, 2xy)$$
is also harmonic.

Answers to problems

A66: $\partial z/\partial x = 5x^4 - 9x^2y^3$, $\partial z/\partial y = 5y^4 - 9x^3y^2$, $\partial^2 z/\partial x^2 = 20x^3 - 18xy^3$, $\partial^2 z/\partial x\partial y = -27x^2y^2$, $\partial^2 z/\partial y^2 = 20y^3 - 18x^3y$. **A67**: See these webnotes[20] for solution **A68**: No **A69**: 0.

A70: $J_{\boldsymbol{x}}\boldsymbol{f} = \begin{bmatrix} y\sin z & x\sin z & xy\cos z \\ y\cos z & x\cos z & -xy\sin z \\ 2x & 2y & 2z \end{bmatrix}$, $J_{(2,1,0)}\boldsymbol{f} = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 2 & 0 \\ 4 & 2 & 0 \end{bmatrix}$. **A74**: $f_x = 0$, $f_y = 0$; f is not differentiable, see these webnotes[21] for solution

[20] http://web.maths.unsw.edu.au/~potapov/2111_2015/Clariaut-Theorem.html
[21] http://web.maths.unsw.edu.au/~potapov/2111_2015/Differentiability-Example-II.html
[22] http://web.maths.unsw.edu.au/~potapov/2111_2015/Differentiability-Example-III.html
[23] http://web.maths.unsw.edu.au/~potapov/2111_2015/Differentiability-of-vector-map.html

A75: $f_x = 1$, $f_y = 1$; f is not differentiable, see these webnotes[22] for solution

A76: $f_x = 0$, $f_y = 0$; see these webnotes[23] for solution

A78: $f_x = 0$, $f_y = 0$; see these webnotes[24] for solution

A79: $\begin{bmatrix} e \\ -1 \end{bmatrix} + \begin{bmatrix} e & -2e \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x-1 \\ y+1 \end{bmatrix}$

A80: ii) .0408

A81: 0.024

A85: $f_x g_t + f_y h_t + f_z k_t$. **A87**: i) $y\,\boldsymbol{i} + x\,\boldsymbol{j}$, $2x\,\boldsymbol{i} + 2y\,\boldsymbol{j}$, $-2y/x^3\,\boldsymbol{i} + 1/x^2\,\boldsymbol{j}$. **A88**: i) $-(\sin r/r)\boldsymbol{r}$, $[(1-\log r)/r^3]\boldsymbol{r}$. **A89**: i) $-155/13$, ii) $10/\sqrt{3}$. **A90**: $(3,-2)$. **A91**: i) $8/5\sqrt{17}$, ii) North East.

A92: i) $\sqrt{3}$, ii) $\pm(2\,\boldsymbol{i} + \boldsymbol{k})$, iii) $\sqrt{5}$.

A93: i) $2\,\boldsymbol{i} + \boldsymbol{j}$, ii) $y = \exp\left[-2.5 \times 10^5/x^2 + 25\right]$.

A94: i) $1 - \dfrac{1}{\sqrt{2}}$, ii) 2π.

A95: See these webnotes[25] for solution **A98**: see these webnotes[26] for solution; see these webnotes[27] for alternative solution **A99**: (i) $\partial^2 f/\partial r^2 = \cos^2\theta\, f_{xx} + \sin 2\theta\, f_{xy} + \sin^2\theta\, f_{yy}$, $\partial^2 f/\partial\theta^2 = r^2(\sin^2\theta\, f_{xx} - \sin 2\theta\, f_{xy} + \cos^2\theta\, f_{yy}) - r(\cos\theta\, f_x + \sin\theta\, f_y)$, $\partial^2 f/\partial r\partial\theta = r/2 \sin 2\theta (f_{yy} - f_{xx}) + r\cos 2\theta\, f_{xy} - \sin\theta\, f_x + \cos\theta\, f_y$; see these webnotes[28] for solution **A100**: $\Delta f = f_{rr} + \frac{1}{r^2} f_{\theta\theta} + \frac{1}{r} f_r$; see these webnotes[29] for solution **A102**: $dg/dt = -\sin t\ \partial f/\partial x + \cos t\ \partial f/\partial y + \partial f/\partial t$. **A103**: $\partial g/\partial u = \partial f/\partial x\, \partial x/\partial u + \partial f/\partial u$.

5 Taylor series, tangent planes, local and global extrema, Lagrange multipliers

[M] – Maple/Gnuplot; [A] – additional/optional problems; [H] – harder problems.

5.1 Taylor series

106: Find the Taylor series, about the point $(1,1)$ for

$$f(x,y) = x^2 + y^2 + xy - x - y.$$

What is the Taylor series for f about the point $(0,0)$?

107: Use a Taylor series to express

$$x(2 + x - 2y^2) + 2y^2$$

in terms of powers of $x-1$ and $y+1$.

108: Without finding the partial derivatives of

$$f(x,y) = e^x \cos y,$$

find the terms up to and including order 4 in the Taylor series for f about $(1,0)$.

5.2 Applications of the gradient

109: Show that the curve

$$\boldsymbol{r}(t) = t^2\,\boldsymbol{i} + t\,\boldsymbol{j} + (5t-4)\boldsymbol{k}$$

is normal to the surface

$$2x^2 + y^2 + 5z^2 = 8$$

at the point $(1,1,1)$.

110: Find the equation for the tangent plane to the surface

$$x^2 + y^2 + z = 9$$

at the point $(1,1,7)$.

111: Given that k is a positive real number, find the equation of the tangent plane to the surface

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{k},$$

at a general point $P(a,b,c)$ lying on the surface. Show that the sum of the intercepts of this plane on the three coordinate axes is independent of the point P.

[24] http://web.maths.unsw.edu.au/~potapov/2111_2015/Differentiability-Example-IV.html
[25] http://web.maths.unsw.edu.au/~potapov/2111_2015/Example-of-directional-derivative.html
[26] http://web.maths.unsw.edu.au/~potapov/2111_2015/f_005fx-and-f_005fy-via-f_005fr-and-f_005ftheta.html
[27] http://web.maths.unsw.edu.au/~potapov/2111_2015/f_005fx-and-f_005fy-via-f_005fr-and-f_005ftheta-II.html
[28] http://web.maths.unsw.edu.au/~potapov/2111_2015/f_005fxx-and-f_005fyy-via-f_005frr-and-f_005ftt-and-f_005fr.html
[29] http://web.maths.unsw.edu.au/~potapov/2111_2015/f_005fxx-and-f_005fyy-via-f_005frr-and-f_005ftt-and-f_005fr.html

112: Show that the tetrahedron bounded by the coordinate axes and the tangent plane at any point P on the surface

$$xyz = 1$$

has a volume which is independent of the point P.

113: Show that if $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function then all the tangent planes to the surface

$$z = y\, f\left(\frac{x}{y}\right)$$

meet in a point.

[H] **114**: Assuming that the equation

$$e^z - y + xz = 0$$

defines z as a function of x and y with $z(1,1) = 0$ and that this function has a Taylor series about $(1,1)$, find the terms up to and including order 2 in this Taylor series and use this to find an approximate value for z when $x = 1.01$ and $y = 0.9$.

5.3 Absolute maximum and minimum

115: Find the greatest and the least values of the function

$$z = x^3 + y^3 - 3xy$$

on the region

$$\{(x,y) \,:\, 0 \le x \le 2,\ -2 \le y \le 2\},$$

along with the points at which the extreme values occur.

116: Find the greatest and the least values of the given function on the given region and the points at which they occur.

i) $z = x^2 + y^2 - xy - y - x$,

$$0 \le x \le 3 \text{ and } 0 \le y \le 3.$$

ii) $z = x^3 + y^3 - 3xy$,

$$0 \le x \le 2 \text{ and } -2 \le y \le 2.$$

117: Find the greatest and least values of

$$f(x,y) = x^2 - y^2$$

on the triangular region T with vertices at

$$(0,2) \text{ and } (\pm 1, -1).$$

5.4 Local maximum and minimum, saddle points

118: Find and classify the critical points of the following functions from $\mathbb{R}^2$ to $\mathbb{R}$.

i) $x^3 - y^3 - 2xy + 4$.

ii) $6x^2 - 2x^3 + 3y^2 + 6xy$.

iii) $(x^2 + y^2)^2 - (x^2 - y^2)$.

119: Verify that the function

$$f(x,y,z) = x^4 + y^4 + z^4 - 4xyz$$

has a stationary point at $(1,1,1)$, and determine the nature of this stationary point by computing the eigenvalues of its Hessian matrix.

5.5 Constrained extrema and Lagrange multipliers

120: Find the maximum and minimum values of

$$f(x,y,z) = x + 2y - 3z$$

on the sphere

$$x^2 + y^2 + z^2 = 14.$$

121: Find the dimensions of a rectangular box, open at the top, which has maximum volume if the surface area is 12 square units.

122: Prove the following inequality

$$\left(\frac{x+y}{2}\right)^n \le \frac{x^n + y^n}{2},$$

for every

$$x, y \ge 0 \text{ and } n \ge 1.$$

Hint: Minimise the function

$$f(x,y) = x^n + y^n$$

subject to constraint

$$x + y = a.$$

123: Find the maximum value of

$$f(x, y, z) = \ln x + 2\ln y + 3\ln z$$

on the part of the sphere

$$x^2 + y^2 + z^2 = r^2$$

lying in the first octant. Deduce that for any positive real numbers a, b, c we have

$$ab^2c^3 \le \frac{(a^2 + b^2 + c^2)^3}{12\sqrt{3}}.$$

124: Find the points on the curve

$$x^2 + xy + y^2 = 2$$

that are closest to the origin.

[H] **125**: If $a > 0$ and $b^2 < ac$, the equation

$$ax^2 + 2bxy + cy^2 = 1$$

describes an ellipse E with centre at the origin. Show that the formula

$$\sqrt{\frac{2}{(a+c) \pm \sqrt{(a-c)^2 + 4b^2}}}$$

gives the distances from the origin to the nearest and furthest points on the curve E. [Hints: First show that if you apply the Lagrange method to $\lambda f - g$ (which is equivalent to using the usual $f + \lambda g$) then the values of the Lagrange multipliers λ for this problem are the eigenvalues of the (symmetric) matrix

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

and hence find expressions for these λs in terms of a, b, c. Observe that if $\boldsymbol{x} = (x, y)$ then the equation for E can be expressed as

$$\boldsymbol{x}^T A \boldsymbol{x} = 1.$$

Show that if $\boldsymbol{x}$ satisfies this equation and is an eigenvector of A then

$$\lambda \boldsymbol{x}^T \boldsymbol{x} = 1.]$$

5.6 Lagrange multipliers, two contraints, non compact

[H] **126**: Prove *the Hölder inequality*

$$ax + by \le (a^p + b^p)^{\frac{1}{p}} \times (x^q + y^q)^{\frac{1}{q}},$$

where

$$x, y, a, b \ge 0 \;\text{ and }\; p, q \ge 1$$

and

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Hint: Maximise the function

$$f(x, y) = ax + by$$

subject to constraint

$$x^q + y^q = c.$$

[H] **127**: A pentagon is made by putting an isosceles triangle on one side of a rectangle. What dimensions will minimise the perimeter of the pentagon for a given area A?

[H] **128**: An open-topped metal water tank with volume $2\,\mathrm{m}^3$ is to be constructed with vertical sides and a right-angled triangle as base. What should be the dimensions of the base to minimise the area of metal used?

129: Find the point closest to the origin on the intersection of the planes

$$x + 2y = 12 \;\text{ and }\; y + z = 6.$$

Answers to problems

A106: $1 + 2(x-1) + 2(y-1) + (x-1)^2 + (y-1)^2 + (x-1)(y-1)$, $-x - y + x^2 + xy + y^2$.

A107: $3 + 2(x-1) + (x-1)^2 + 4(x-1)(y+1) - 2(x-1)(y+1)^2$. **A108**: $e\Big[1 + (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{2}y^2 + \frac{1}{6}(x-1)^3 - \frac{1}{2}(x-1)y^2 + \frac{1}{24}(x-1)^4 - \frac{1}{4}(x-1)^2y^2 + \frac{1}{24}y^4\Big]$. **A110**: $2x + 2y + z = 11$. **A113**: The point is the origin. **A114**:

$\frac{1}{2}(y-1) - \frac{1}{4}(x-1)(y-1) - \frac{1}{16}(y-1)^2$, $z = -0.050375$ (the correct value is $z = -0.0503719....$) **A115**: min -8 at $(0,-2)$, max $8+4\sqrt{2}$ at $(2,-\sqrt{2})$. **A116**: i) min -1 at $(1,1)$, max 6 at $(3,0)$ and $(0,3)$. ii) min -8 at $(0,-2)$, max $8+4\sqrt{2}$ at $(2,-\sqrt{2})$. **A117**: min -4 at $(0,2)$, max $1/2$ at $(\pm 3/4, -1/4)$. **A118**: i) S.P. $(0,0)$, max $(-2/3, 2/3)$ ii) S.P. $(1,-1)$, min $(0,0)$ iii) S.P. $(0,0)$, min $\left(\pm 1/\sqrt{2}, 0\right)$.

A119: Eigenvalues are 4, 16, 16 so $(1,1,1)$ is a minimum. **A120**: max $= 14$, min $= -14$. **A121**: $2 \times 2 \times 1$. **A123**: $\ln \dfrac{r^6}{12\sqrt{3}}$. **A124**: $\left(\pm\sqrt{2/3}, \pm\sqrt{2/3}\right)$. **A126**: See these webnotes[30] for solution **A127**: The triangle should have height $h = \sqrt{A/(6+3\sqrt{3})}$ and base $2h\sqrt{3}$. The remaining side of the rectangle should have length $(1+\sqrt{3})h$. **A128**: The base should have two sides of length $(8+4\sqrt{2})^{1/3}$. **A129**: $(2,5,1)$

6 Implicit differentiation, Implicit and Inverse Function Theorems

6.1 Implicit differentiation

130: Assuming that the equations

$$x^2 + yu + xv + w = 0,$$
$$x + y + uvw + 1 = 0$$

define x and y as differentiable functions of u, v and w, find

$$\frac{\partial x}{\partial u} \text{ and } \frac{\partial y}{\partial u},$$

where $x = 1$, $y = -1$, $u = 1$, $v = 1$ and $w = -1$.

131: Assuming that the equations

$$x^2 + y^2 + u^2 + v^2 = 1,$$
$$x^2 + 2y^2 - u^2 + v^2 = 1,$$

define x and y as differentiable functions of u and v, find (in terms of u, v, x, y) formulae for

$$\frac{\partial x}{\partial u},\ \frac{\partial y}{\partial u},\ \frac{\partial^2 x}{\partial u^2} \text{ and } \frac{\partial^2 y}{\partial u^2}.$$

132: Assuming that the equation

$$F(x,y,z) = 0$$

defines z implicitly as a differentiable function of x and y and that

$$F_{xz} = F_{zx},$$

show that

$$\frac{\partial^2 z}{\partial x^2} = \frac{-(F_z)^2 F_{xx} + 2F_z F_x F_{xz} - (F_x)^2 F_{zz}}{(F_z)^3}.$$

6.2 Inverse and implicit function theorems

133: For $x \in \mathbb{R}$ and $y > 0$, let $\boldsymbol{f}$ be defined by

$$\boldsymbol{f}(x,y) = \begin{bmatrix} x^2 + \ln y \\ x^4 + y^3 \end{bmatrix}.$$

Let $\boldsymbol{x}_0 = (1,1)$.

i) Show that $\boldsymbol{f}$ has a local C^1 inverse near near $\boldsymbol{f}(\boldsymbol{x}_0)$. Find the Jacobian matrix at $\boldsymbol{f}(\boldsymbol{x}_0)$ for this local inverse.

ii) Find the best affine approximation to $\boldsymbol{f}^{-1}$ near $\boldsymbol{f}(\boldsymbol{x}_0)$. Use this to find an approximate solution to the pair of equations $x^2 + \ln y = 1.05$, $x^4 + y^3 = 1.90$.

134: Let $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\boldsymbol{f}(x,y) = \begin{bmatrix} x^3 - 2xy^2 \\ x + y \end{bmatrix}$$

and let $\boldsymbol{x}_0 = (1,1)$.

i) Show that for every $\boldsymbol{b}$ in some neighbourhood of $\boldsymbol{f}(\boldsymbol{x}_0)$ the non-linear system of equations $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{b}$ has a solution for $\boldsymbol{x}$ in terms of $\boldsymbol{b}$.

ii) Is the solution in (i) unique for every $\boldsymbol{b}$?

iii) Find the Jacobian matrix at $\boldsymbol{f}(\boldsymbol{x})$ for the function which gives $\boldsymbol{x}$ in terms of $\boldsymbol{b}$ and use this to find an approximate solution to the pair of equations $x^3 - 2xy^2 = -1.01$, $x + y = 2.01$.

[30] http://web.maths.unsw.edu.au/~potapov/2111_2015/Holder-inequality.html

135: For $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\boldsymbol{f}(x,y) = \begin{bmatrix} e^x \cos y \\ e^x \sin y \end{bmatrix},$$

show that $\boldsymbol{f}$ has a local C^1 inverse near every point in $\mathbb{R}^2$ but does not have a global inverse $\mathbb{R}^2 \to \mathbb{R}^2$.

136: Show that in some neighbourhood Ω of $(x,y) = (1,1)$ in $\mathbb{R}^2$ the equations

$$\begin{aligned} x - y + 2u + v + 2w - 1 &= 0, \\ x - 2y - uv - w + 1 &= 0, \\ x^2 + y + (u+v)w &= 0, \end{aligned}$$

define a differentiable function $\boldsymbol{f} : \Omega \to \mathbb{R}^3$ taking (x,y) to (u,v,w) with $\boldsymbol{f}(1,1) = (1,1,-1)$. Find $J_{(1,1)}\boldsymbol{f}$.

137: Show that in some neighbourhood of $(x,y,u,v) = (0,1,2,1)$ in $\mathbb{R}^4$ the non-linear system

$$\begin{aligned} e^{xyu} + yuv + x - 3 &= 0, \\ \ln yv + xu^3 v - x^3 u &= 0, \end{aligned}$$

has a unique solution $\big(x, y, u(x,y), v(x,y)\big)$ for every (x,y) and this solution depends differentiably on (x,y). That is, show that there is a C^1 function $f : \mathbb{R}^2 \to \mathbb{R}^2$ so that for (x,y) near $(0,1)$, and

$$\begin{pmatrix} u \\ v \end{pmatrix} = f\begin{pmatrix} x \\ y \end{pmatrix}.$$

(x,y,u,v) satisfy the non-linear equations. Find the best affine approximation to f at $(0,1)$ and also for its inverse and use these to find an approximate solution with $x = 0.01$ and $y = 1.05$ and another one with $u = 1.99$ and $v = 1.05$.

138: This question is about whether or not the equations

$$\begin{aligned} xyt + \sin xyt &= 0, \\ x + y + t &= 0, \end{aligned}$$

define x and y as functions of t.

i) Show that the equations do define x and y uniquely as differentiable functions of t near $(x,y,t) = (0,1,-1)$.

ii) Show that the assumptions of the Implicit Function Theorem are not satisfied at $(x,y,t) = (0,0,0)$.

iii)* To confirm that there is no neighbourhood of $(0,0,0)$ in which the equations define x and y uniquely as functions of t, find two straight lines through the origin in (x,y,t)-space on which all points satisfy the given equations.

Answers to problems

A130: 0, 1.
A131: $-3u/x,\ 2u/y,\ -(3x^2+9u^2)/x^3,\ (2y^2 - 4u^2)/y^3$. **A133**: $x = 1.125$, $y = 0.8$.
A134: (ii) there will be a unique solution near $\boldsymbol{x}_0$, but other solutions also exist.
(iii) $x = 1.006$, $y = 1.004$. **A136**: $\dfrac{1}{3}\begin{bmatrix} -5 & 12 \\ 9 & -15 \\ -1 & -3 \end{bmatrix}$. **A137**: $(0.01, 1.05, 2.13, 0.87)$, $(-0.01/13, 12.43/13, 1.99, 1.05)$. **A138**: The lines $(x,y,t) = \lambda(1,0,-1)$, $\lambda \in \mathbb{R}$, and $(x,y,t) = \lambda(0,1,-1)$, $\lambda \in \mathbb{R}$.

7 Integration

7.1 Double integrals, definition

139: Let R be the rectangle

$$\Big\{ (x, y) : \quad 0 \le x \le 1,\ 0 \le y \le 2 \Big\}$$

and P_n be the partition of $[0, 1]$ given by

$$P_n = \Big\{ \frac{k}{n} : \quad 0 \le k \le n \Big\}$$

and let Q_n be the partition of $[0, 2]$ given by

$$Q_n = \Big\{ \frac{k}{n} : \quad 0 \le k \le 2n \Big\}.$$

Calculate the Riemann sum for $f(x, y) = x^2y^2$ with respect to these partitions, using the lower left corner of each sub-rectangle as the point at which f is evaluated. Find the limit of this Riemann sum as $n \to \infty$.

7.2 Double integrals via repeated integrals

140: Evaluate the integral of $f(x, y) = xy$ over the region bounded by the x-axis, the line $x = 2a$ and the parabola $x^2 = 4ay$.

141: For each of the following integrals, sketch the region of integration, reverse the order of integration and evaluate the integral.

i) $\displaystyle\int_0^{\pi/2} \int_y^{\pi/2} \frac{\sin x}{x}\, dx\, dy,$

ii) $\displaystyle\int_0^1 \int_x^1 \frac{y^\lambda}{x^2 + y^2}\, dy\, dx$, where $\lambda > 0$,

iii) $\displaystyle\int_0^1 \int_x^{x^{1/3}} \sqrt{1 - y^4}\, dy\, dx,$

iv) $\displaystyle\int_1^2 \int_x^{x^3} \sqrt{x/y}\, dy\, dx + \int_2^8 \int_x^8 \sqrt{x/y}\, dy\, dx.$

7.3 Triple integrals via repated integrals

142: Evaluate

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx$$

and interpret the answer as a volume. More generally, find the volume of the tetrahedron bounded by the planes

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,\ x = 0,\ y = 0,\ z = 0.$$

143: Evaluate

$$\iiint_S x^2\ dx\, dy\, dz,$$

where S is the region bounded by

$$4x^2 + y^2 = 4, \quad z + x = 2 \ \text{ and } \ z = 0.$$

144: Find the volume of the region enclosed between the parabolic cylinder $z = y^2$ and the elliptic paraboloid $z = 16 - 4x^2 - y^2$.

7.4 Leibniz' rule

145: Given that

$$\int_0^\pi \frac{dx}{t - \cos x} = \frac{\pi}{\sqrt{t^2 - 1}}, \quad t > 1,$$

evaluate

$$\int_0^\pi \frac{dx}{(2 - \cos x)^2}.$$

146: i) Show that if $x(t)$ satisfies the integral equation

$$x(t) = a + bt + \int_0^t (t - s) f\big(x(s)\big)\, ds$$

then $x(t)$ is a solution to the initial value problem

$$x''(t) = f(x(t))$$

for $t > 0$, with $x(0) = a$, $x'(0) = b$.

ii) Prove the converse of the result in part (i).
[Hint: You will need to do a change of order of integration in a double integral.]

147: i) Prove that if the mixed derivatives f_{xy} and f_{yx} are continuous then they are equal. *Hint:* Use Leibniz' rule and both versions of the Fundamental Theorem of Calculus to find

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}\int_c^y f_y(x,t)\,dt$$

in two different ways.

ii) Re-examine your proof in part (i) to confirm that to prove f_{xy} exists and equals f_{yx} on

$$R = [a,b]\times[c,d]$$

you need only assume that f_y and f_{yx} exist and are continuous on R and that $f_x(x,c)$ exists for

$$a \le x \le b.$$

iii) Using Fubini's theorem (as a theorem about equality of iterated integrals with different orders of integration) and both versions of the Fundamental Theorem of Calculus, prove Leibniz' Rule. *Hint:* Find

$$\frac{d}{dx}\int_c^d \left[\int_a^x f_x(t,y)\,dt\right]\,dy$$

in two different ways.

7.5 Double integrals in polar coordinates

148: Use polar co-ordinates to evaluate

$$\int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \sqrt{x^2+y^2}\,dx\,dy.$$

149: A plane region R is determined by the inequalities

$$y \ge 0 \quad\text{and}\quad y \ge -x$$

and

$$x^2+y^2 \le 3\sqrt{x^2+y^2} - 3x.$$

Sketch the region and find its area.

150: i) Evaluate

$$\int_0^\infty \int_0^\infty \exp(-x^2-y^2)\,dx\,dy.$$

[Hint: Evaluate

$$\iint_R e^{-x^2-y^2}\,dx\,dy,$$

where R is a quarter circle of radius M. Then let $M\to\infty$. Be aware that this is an improper integral and we really should say how it is defined!]

ii) Use the result of part (i) to evaluate

$$\int_0^\infty e^{-x^2}\,dx$$

and hence evaluate

$$\int_0^\infty t^{-1/2}e^{-t}\,dt.$$

[H] **151**: For $r > 0$ and n a positive integer, let $V_n(r)$ be the volume of the n-dimensional ball

$$B_n(r) = \{\, \boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| \le r \,\}.$$

i) Show that, for $n \ge 3$,

$$V_n(a) = \int_0^{2\pi} d\theta \int_0^a V_{n-2}\left(\sqrt{a^2-r^2}\right)\, r\,dr.$$

ii) Assuming that $V_n(r) = k_n r^n$, where k_n is independent of r, show that the sequence $\{k_n\}$ satisfies the recurrence relation

$$k_n = (2\pi/n)\,k_{n-2},$$

for $n \ge 3$, with $k_1 = 2$ and $k_2 = \pi$.

iii) The Gamma function is defined for real $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

Prove that

$$\Gamma(x+1) = x\Gamma(x)$$

and deduce that the sequence

$$k_n = \frac{\pi^{n/2}}{\Gamma(\frac{1}{2}n+1)}.$$

satisfies the recurrence relation and initial conditions in part (ii). [Note that $\Gamma(\frac{1}{2})$ was evaluated in the previous question.]

iv) Deduce that

$$V_n(r) = \frac{\pi^{n/2}r^n}{\Gamma(\frac{1}{2}n+1)}, \quad n \ge 1,$$

and in particular

$$V_{2n}(r) = \pi^n r^{2n}/n!$$

for all positive integers n.

7.6 Cylindrical and spherical coordinates

152: Find the volume of the solid enclosed between the spheres

$$x^2 + y^2 + z^2 = 4 \text{ and } x^2 + y^2 + z^2 = 4z.$$

153: Find the volume inside the cone

$$z + 2 = \sqrt{x^2 + y^2}$$

between the planes $z = 0$ and $z = 1$. [Hint: The innermost integration should be with respect to r.]

154: A cylindrical hole 10cm long and 6cm in diameter is drilled through the centre of a steel ball with the axis of the hole being a diameter of the ball. What is volume of steel left in the resulting solid?

[H] **155**: Let $a > 0$. Find the volume of the region specified by

$$x^2 + y^2 \leq a^2, \quad x^2 + z^2 \leq a^2 \text{ and } y^2 + z^2 \leq a^2.$$

156: Express as a triple integral using

i) Cartesian co-ordinates

ii) cylindrical co-ordinates

iii) spherical co-ordinates

the volume of the region above the cone

$$z = \sqrt{x^2 + y^2}$$

and inside the sphere

$$x^2 + y^2 + z^2 = 2az, \quad (a > 0).$$

Evaluate the volume by spherical co-ordinates.

157: A solid occupies the region Ω bounded above by the sphere

$$x^2 + y^2 + z^2 = 9$$

and below by the cone

$$x^2 + y^2 = z^2$$

and its density function is

$$\delta(x, y, z) = 2 + x^2 + y^2 + z^2.$$

Use spherical coordinates to find the moment of inertia of this solid about the z-axis, i.e. calculate

$$\iiint_\Omega (x^2 + y^2)\delta(x, y, z)\, dx\, dy\, dz.$$

158: Use spherical coordinates to find the volume enclosed by the surface

$$(\sqrt{x^2 + y^2} - 1)^2 + z^2 = 1.$$

7.7 Other changes of variable

159: Evaluate

$$\iint_{\Omega'} x^2 y^2\, dx\, dy,$$

where Ω' is the bounded portion of the first quadrant lying between the two hyperbolas

$$xy = 1 \text{ and } xy = 2$$

and the two straight lines

$$y = x \text{ and } y = 4x.$$

160: Let Ω' be the region in the first quadrant bounded by the hyperbolas

$$x^2 - 2y^2 = 1, \quad x^2 - 2y^2 = 3$$

and

$$xy = 1, \quad xy = 2.$$

Let

$$u = x^2 - 2y^2 \text{ and } v = xy.$$

Sketch the region Ω' in the (x, y)-plane and the region Ω in the (u, v)-plane that corresponds to Ω'. Hence evaluate

$$\iint_{\Omega'} (x^2 - 2y^2)x^2y^2(2x^2 + 4y^2)\, dx\, dy.$$

[Hint: You will not need to solve for x and y in terms of u and v because the Jacobian cancels with the awkward factor in the integrand.]

161: Integrate the function $(xy)^{-1}$ over the region Ω' bounded by the four circles

$$x^2 + y^2 = ax \text{ and } x^2 + y^2 = a'x$$

and

$$x^2 + y^2 = by \text{ and } x^2 + y^2 = b'y,$$

where

$$a < a' \quad \text{and} \quad b < b'.$$

[Hint: $u = (x^2 + y^2)/x$ is constant on two of the circles.]

162: i) By means of the formula for the volume of an n-dimensional ball, find the volume enclosed by the hyperellipsoid

$$\left\{ \boldsymbol{x} \in \mathbb{R}^n : \sum_{i=1}^{n} (x_i/a_i)^2 = 1 \right\},$$

where $a_i > 0$ for $i = 1, 2, \ldots, n$.

ii) Use spherical coordinates to evaluate

$$\iiint_B z^2 \, dx \, dy \, dz,$$

where B is the unit ball centred at the origin in $\mathbb{R}^3$. What is the value of

$$\iiint_B y^2 \, dx \, dy \, dz?$$

iii) A solid occupies the region Ω in $\mathbb{R}^3$ enclosed by the ellipsoid

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1,$$

where a, b, c are all positive. Find the moment of inertia of this solid about the x-axis if the density is a constant k, i.e. evaluate

$$\iiint_\Omega k(y^2 + z^2) \, dx \, dy \, dz.$$

[Hint: use the results of part (ii).]

7.8 Mass and centre of mass

163: A lamina is bounded by the curves

$$x^2 + y^2 = 1 \quad \text{and} \quad 2x + y = 1$$

in the first quadrant. Find the mass of this lamina if it has density x^2y gm/unit^2.

164: A plane lamina is bounded by the curves

$$y = x^2 \quad \text{and} \quad y = x^3$$

in the first quadrant. Its density function is

$$\delta(x, y) = \sqrt{xy}.$$

Find its centre of mass.

165: Find the centre of mass of the region V in $\mathbb{R}^3$ where

$$x^2 + y^2 \geq 1, \quad x^2 + y^2 + z^2 \leq 2 \quad \text{and} \quad y \geq 0,$$

given that the density function for the region is

$$\delta(x, y, z) = (x^2 + y^2)^{-1/2}.$$

Answers to problems

A139: 8/9
A140: $a^4/3$. **A141**: i) 1 ii) $\pi/4\lambda$ iii) $\pi/8 - 1/6$ iv) 49/3 **A142**: 1/6; $\frac{abc}{6}$.
A143: π. **A144**: $32\pi\sqrt{2}$. **A145**: $2\pi/3^{3/2}$.
A148: 32/9.
A149: $9(9\pi - 8\sqrt{2} - 2)/16$. **A150**: i) $\pi/4$, ii) $\sqrt{\pi}/2$, $\sqrt{\pi}$. **A152**: $10\pi/3$.
A153: $19\pi/3$. **A154**: $256\pi/3$. **A155**: $8a^3(2 - \sqrt{2})$. **A156**: πa^3. **A157**: $3^4\pi(8 - 5\sqrt{2})(2/5 + 9/7)/2 \approx 199.24$. **A158**: $2\pi^2$. **A159**: $(7/3)\ln 2$.

A160: 28/3. **A161**: $\ln(a'/a)\ln(b'/b)$. **A162**: i) $\dfrac{\pi^{n/2}}{\Gamma(\frac{1}{2}n + 1)} \prod_{i=1}^{n} a_i$, ii) both equal $\dfrac{4\pi}{15}$, iii) $\dfrac{4\pi kabc(b^2 + c^2)}{15}$ **A163**: 31/480 **A164**: $(54/77, 6/13)$. **A165**: $\big(0, 8/(3\pi(\pi - 2)), 0\big)$.