MATH2111 Higher Several Variable Calculus
Integration

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Semester 1, 2014

Riemann Integral (one variable)

A set of points $\mathcal{P} = \{x_0, x_1, x_2, \ldots, x_n\}$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, is called a partition of $[a, b]$. For a function $f : [a, b] \to \mathbb{R}$, the upper and lower Riemann sums of f with respect to $\mathcal{P}$ are

$$\underline{S}_{\mathcal{P}}(f) = \sum_{k=1}^{n} \underline{f}_k \Delta x_k \quad \text{and} \quad \overline{S}_{\mathcal{P}}(f) = \sum_{k=1}^{n} \overline{f}_k \Delta x_k$$

where $\underline{f}_k$ and $\overline{f}_k$ are the infimum and supremum of f on $[x_{k-1}, x_k]$ and $\Delta x_k = x_k - x_{k-1}$.

For a bounded function $f : [a, b] \to \mathbb{R}$, if there exists a unique number I such that

$$\underline{S}_{\mathcal{P}}(f) \le I \le \overline{S}_{\mathcal{P}}(f)$$

for every partition $\mathcal{P}$ of $[a, b]$, then f is Riemann integrable on $[a, b]$ and

$$I = \int_{[a,b]} f = \int_a^b f(x)dx.$$

First consider $f : R \to \mathbb{R}$, where $R = [a, b] \times [c, d]$ is a rectangle in $\mathbb{R}^2$.

Let

$$P_1 = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$$

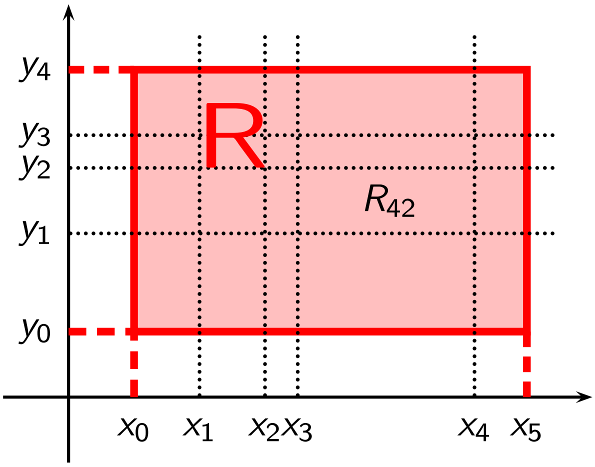
be a partition of $[a, b]$ and

$$P_2 = \{c = y_0, y_1, y_2, \ldots, y_m = d\}$$

a partition of $[c, d]$.

Then R is the union of the mn subrectangles

$$R_{jk} = [x_{j-1}, x_j] \times [y_{k-1}, y_k].$$



The upper and lower Riemann sums of f with respect to these partitions are

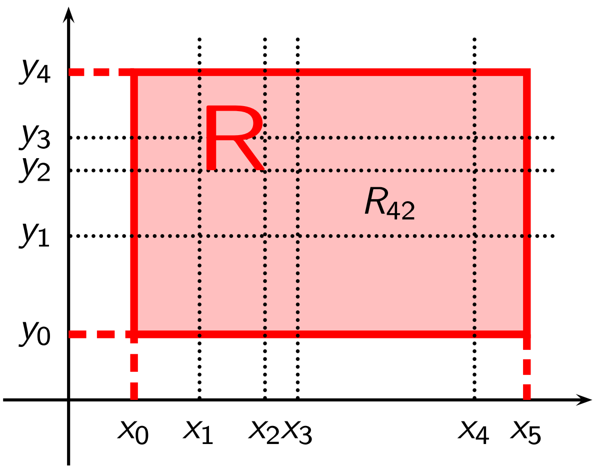
$$\underline{S}_{\mathcal{P}_1,\mathcal{P}_2}(f) = \sum_{j,k} \underline{f}_{jk} \Delta x_j \Delta y_k$$

and

$$\overline{S}_{\mathcal{P}_1,\mathcal{P}_2}(f) = \sum_{j,k} \overline{f}_{jk} \Delta x_j \Delta y_k.$$

The sums are over all pairs (j, k) with

$$1 \le j \le n \quad \text{and} \quad 1 \le k \le m.$$



The numbers $\underline{f}_{jk}$ and $\overline{f}_{jk}$ are the infimum and supremum of f on R_{jk} and Δx_j and Δy_k its width and height.

Integration

Definition (Riemann integral)

For a bounded function $f : R \to \mathbb{R}$, if there exists a unique number I such that

$$\underline{S}_{\mathcal{P}_1,\mathcal{P}_2}(f) \leq I \leq \overline{S}_{\mathcal{P}_1,\mathcal{P}_2}(f)$$

for every pair of partitions $\mathcal{P}_1, \mathcal{P}_2$ of R, then f is Riemann integrable on R and

$$I = \iint_R f = \iint_R f(x, y)\, dA.$$

I is called the Riemann integral of f over R.

Riemann Integral

For a function of one variable, the Riemann integral is interpreted as the (signed) area bounded by the graph $y = f(x)$ and the x-axis over the interval $[a, b]$. For a function of two variables, $\iint_R f$ is the (signed) volume bounded by the graph $z = f(x, y)$ and the xy-plane over the rectangle R.

Properties:

If f and g are integrable on R,

(1) Linearity: $\displaystyle\iint_R \alpha f + \beta g = \alpha \iint_R f + \beta \iint_R g, \qquad \alpha, \beta \in \mathbb{R}.$

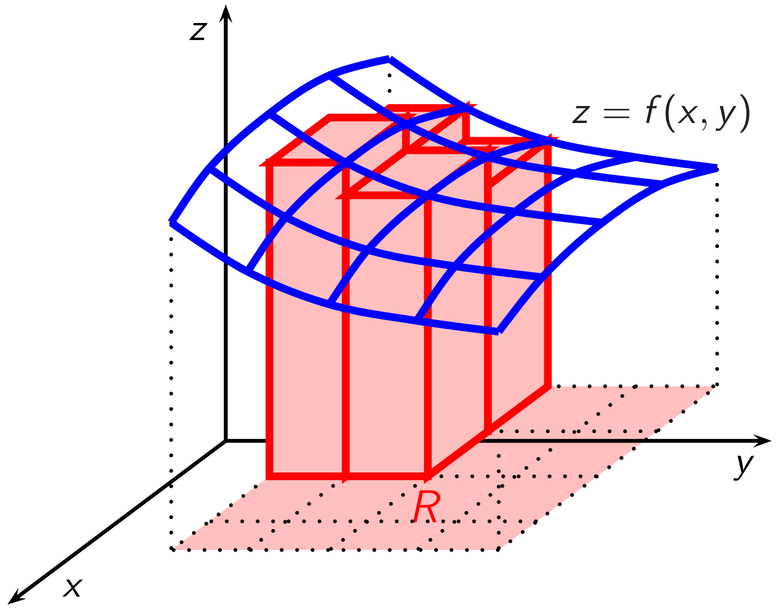
(2) Positivity (monotonicity): If $f(\mathbf{x}) \leq g(\mathbf{x})$ for all $\mathbf{x} \in R$ then $\displaystyle\iint_R f \leq \iint_R g.$

(3) $\displaystyle\left|\iint_R f\right| \leq \iint_R |f|.$

(4) If $R = R_1 \cup R_2$ and (interior R_1)$\cap$(interior R_2) $= \emptyset$ then

$$\iint_R f = \iint_{R_1} f + \iint_{R_2} f.$$

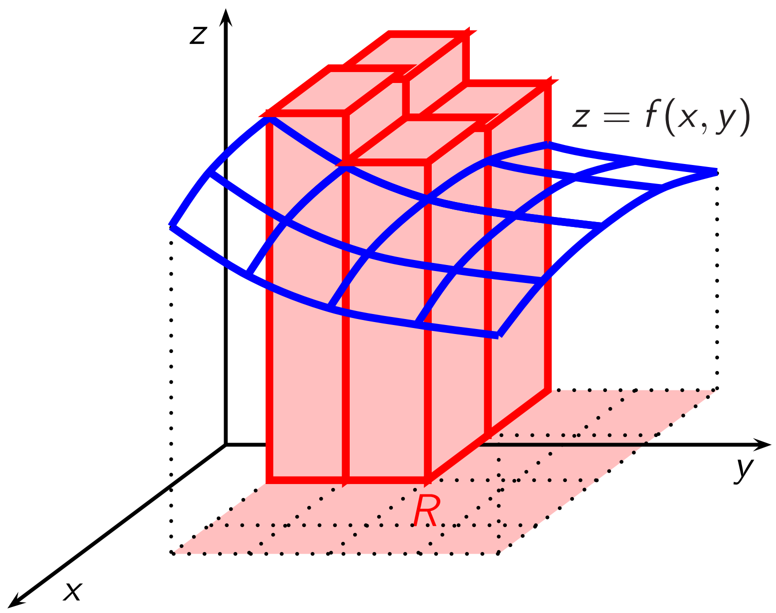
Lower sum



A lower sum of f over R.

$$\underline{S}_{\mathcal{P}_1,\mathcal{P}_2}(f) = \sum_{j,k} \underline{f}_{jk} \Delta x_j \Delta y_k .$$

Upper sum



An upper sum of f over R.

$$\overline{S}_{\mathcal{P}_1,\mathcal{P}_2}(f) = \sum_{j,k} \overline{f}_{jk} \Delta x_j \Delta y_k .$$

Fubini's theorem on rectangles

Theorem (Fubini's Theorem (version 1))

Let $f : R \to \mathbb{R}$ be continuous on a rectangular domain $R = [a, b] \times [c, d]$. Then

$$\int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy = \iint_R f.$$

Note that

$$\int_a^b \int_c^d f(x, y) dy dx$$

means

$$\int_a^b \left(\int_c^d f(x, y) dy \right) dx.$$

The integral inside the brackets is, for each x, a one variable integral.

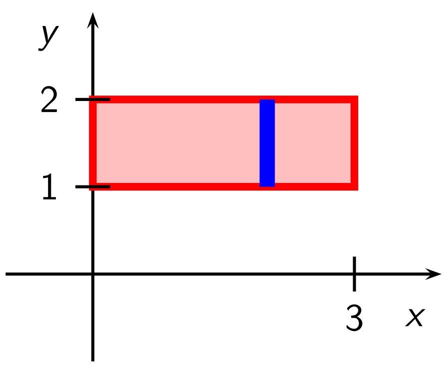
Fubini's theorem on rectangles

Example: Calculate the integral of $f : \mathbb{R}^2 \to \mathbb{R}$, where $f(x, y) = xy^3$, over the rectanglur region $R = [0, 3] \times [1, 2]$.

Fubini's theorem for rectangular regions says gives two ways of calculating this integral.

Let's integrate first with respect to y.

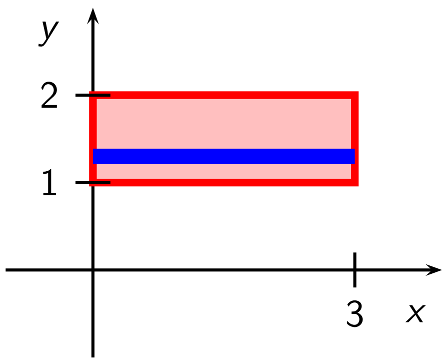
$$\begin{aligned}
\int_R f &= \int_0^3 \int_1^2 xy^3 dy \; dx = \int_0^3 \left[\frac{1}{4} xy^4 \right]_1^2 dx \\
&= \int_0^3 \frac{15}{4} x \; dx \\
&= \frac{15}{4} \left[\frac{1}{2} x^2 \right]_0^3 \\
&= \frac{135}{8}.
\end{aligned}$$



Fubini's theorem on rectangles

We can also integrate first with respect to x.

$$\begin{aligned}\int_R f &= \int_1^2 \int_0^3 xy^3 dx\ dy = \int_1^2 \left[\frac{1}{2}x^2y^3\right]_0^3 dy \\ &= \int_1^2 \frac{9}{2}y^3\ dy \\ &= \frac{9}{2}\left[\frac{1}{4}y^4\right]_1^2 \\ &= \frac{135}{8}.\end{aligned}$$



In this case we also have

$$\int_R f = \int_1^2 \int_0^3 xy^3 dxdy = \int_1^2 y^3 \int_0^3 xdx\ dy = \left(\int_1^2 y^3\ dy\right)\left(\int_0^3 x\ dx\right).$$

Fubini's theorem on rectangles

Fubini's theorem is essentially the same as the method of slicing for calculating volumes from first year. However, proving the iterated integrals give the same number for the volume as the definition involves some subtlety.

The lower Riemann sum can be written as a double sum.

$$\underline{S}_{\mathcal{P}_1,\mathcal{P}_2}(f) = \sum_{j,k} \underline{f}_{j,k}\Delta x_j \Delta y_k = \sum_{k=1}^{m}\left(\sum_{j=1}^{n} \underline{f}_{j,k}\Delta x_j\right)\Delta y_k$$

and it is tempting to consider the sum in the brackets as a one variable Riemann lower sum of $f(x,y)$ for a fixed value of y. However, $\underline{f}_{j,k}$ is not necessarily an infimum for $f(x,y)$ as a function of x for any particular fixed value of y.

One way around this problem is to use the continuity of f to show that $\sum_{j=1}^{n} \underline{f}_{j,k}\Delta x_j$ can be made to be within ε of the one variable lower sum of $f(x, y_k)$ by requiring the spacing in the partition $\mathcal{P}_2$ to be sufficiently small.

A more eligant approach can be found on pages 65–70 of the Internet Supplement to Marsden and Tromba 5th Edition.

Riemann integral over more general regions

Theorem (Integrability of bounded functions)

Let $f : R \to \mathbb{R}$ be a bounded function on the rectangle R and suppose that the set of points where f is discontinuous lies on a finite union of graphs of continuous functions. Then f is integrable over R.

The proof of this theorem is exercises 4, 5 and 6 from the Internet Supplement to Marsden and Tromba. The essence of the proof is that the contribution to the upper and lower Riemann sums from subrectangles containing the the lines of discontinuity can be made arbitrarily small by taking the width of subintervals in the partitions to be small enough.

Riemann integral over more general regions

Theorem (Fubini's Theorem (version 2))

Let $f : R \to \mathbb{R}$ be bounded on a rectangular domain $R = [a, b] \times [c, d]$ with the discontinuities of f confined to a finite union of graphs of continuous functions. If the integral $\int_c^d f(x, y)dy$ exists for each $x \in [a, b]$, then

$$\iint_R f = \int_a^b \left(\int_c^d f(x, y)dy \right) dx.$$

Similarly, if the integral $\int_a^b f(x, y)dx$ exists for each $y \in [c, d]$, then

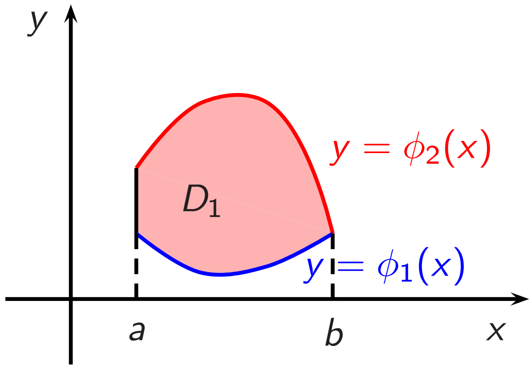
$$\iint_R f = \int_c^d \left(\int_a^b f(x, y)dx \right) dy.$$

Since f is not continuous there is no guarantee that $\int_c^d f(x, y)dy$ exists for each x or that $\int_a^b f(x, y)dx$ exists for each y.

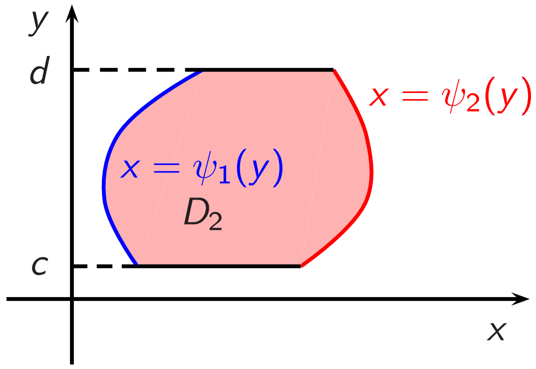
Riemann integral over more general regions

An elementary region is a region of the type illustrated below.

A y-simple region.



An x-simple region.



$$D_1 = \{(x,y) : x \in [a,b] \text{ and } \phi_1(x) \le y \le \phi_2(x)\}$$

where ϕ_1 and ϕ_2 are continuous functions from $[a,b]$ to $\mathbb{R}$.

$$D_2 = \{(x,y) : y \in [c,d] \text{ and } \psi_1(y) \le x \le \psi_2(y)\}$$

where ψ_1 and ψ_2 are continuous functions from $[c,d]$ to $\mathbb{R}$.

Riemann integral over more general regions

Suppose D is an elementary region, R a rectangle containing D and f a function from D to $\mathbb{R}$. First extend f to a function defined on all of R by

$$f^*(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \text{ and } (x,y) \in R. \end{cases}$$

Then f is integrable on D if f^* is integrable on R and

$$\iint_D f = \iint_R f^*.$$

If f is continuous except perhaps on a set of points made from the graphs of continuous functions, then f^* also has this property and so the second version of Fubini's theorem gives us a way to calculate the integral of f over D as an iterated integral.

Iterated integrals for elementary regions

Suppose D_1 is a y-simple region contained in $R = [a,b] \times [c,d]$ and bounded by $x = a$, $x = b$, $y = \phi_1(x)$ and $y = \phi_2(x)$, and $f : D_1 \to \mathbb{R}$ is such that f^* satisfies the conditions of Fubini's Theorem (version 2). Then

$$\iint_{D_1} f = \iint_R f^* = \int_a^b \int_c^d f^*(x,y)dydx,$$

but since $f^*(x,y) = 0$ for $y < \phi_1(x)$ or $y > \phi_2(x)$,

$$\int_c^d f^*(x,y)dy = \int_{\phi_1(x)}^{\phi_2(x)} f^*(x,y)dy = \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy.$$

and hence

$$\iint_{D_1} f = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dydx.$$

A similar result hold for integrals of x-simple regions like D_2.

$$\iint_{D_2} f = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dxdy.$$

Notes

The properties of the integral over rectangular regions also apply to more general regions.

The area of a region D of $\mathbb{R}^2$ is given by $\iint_D 1$. (That is, the integral of the function $f(x,y) = 1$ over D.)

Everything that we have done for integrals in $\mathbb{R}^2$ extends to $\mathbb{R}^n$.

The volume of a region D of $\mathbb{R}^3$ is given by $\iint_D 1$. (That is, the integral of the function $f(x,y,z) = 1$ over D.)

Fubini's theorem

(a) Find the integral of $f(x, y) = x^2y$ over the triangular region Ω with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$.

(b) For each of the following two integrals, find the region over which the integral is calculated and reverse the order of integration.

(i)
$$I_1 = \int_0^1 \int_0^{2-2x} f(x, y)dydx.$$

(ii)
$$I_2 = \int_{-1}^1 \int_{x-1}^{\sqrt{1-x^2}} f(x, y)dydx.$$

Fubini's theorem

(a) Find the integral of $f(x, y) = x^2y$ over the triangular region Ω with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$.

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$$I_2 = \int_{-1}^1 \int_{x-1}^{\sqrt{1-x^2}} f(x, y)dydx.$$

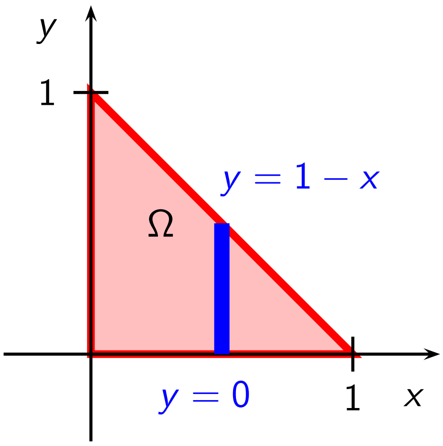
Fubini's theorem examples

Example (a): Find the integral of $f(x,y) = x^2y$ over the triangular region Ω with vertices $(0,0)$, $(0,1)$ and $(1,0)$.

Fubini's theorem gives two ways of calculating this integral.

Let's integrate with respect to y first.

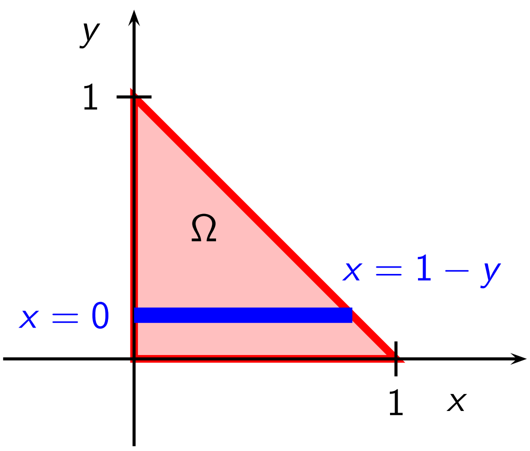
$$\begin{aligned}\int_\Omega f &= \int_0^1 \int_0^{1-x} x^2y\, dy\, dx = \int_0^1 \left[\frac{1}{2}x^2y^2\right]_0^{1-x} dx\\ &= \int_0^1 \frac{1}{2}x^2(1-x)^2\, dx\\ &= \frac{1}{2}\int_0^1 x^2 - 2x^3 + x^4\, dx\\ &= \left[\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_0^1\\ &= \frac{1}{60}.\end{aligned}$$



Fubini's theorem examples

Now check this by integrating with respect to x first.

$$\begin{aligned}\int_\Omega f &= \int_0^1 \int_0^{1-y} x^2y\, dx\, dy\\ &= \int_0^1 \left[\frac{1}{3}x^3y\right]_0^{1-y} dy\\ &= \int_0^1 \frac{1}{3}(1-y)^3y\, dy\\ &= \frac{1}{3}\int_0^1 y - 3y^2 + 3y^3 - y^4\, dy\\ &= \frac{1}{3}\left[\frac{1}{2}y^2 - y^3 + \frac{3}{4}y^4 - \frac{1}{5}y^5\right]_0^1\\ &= \frac{1}{60}.\end{aligned}$$



Fubini's theorem examples

Example (b) (i): Find the region over which the integral

$$I_1 = \int_0^1 \int_0^{2-2x} f(x,y)\; dydx$$

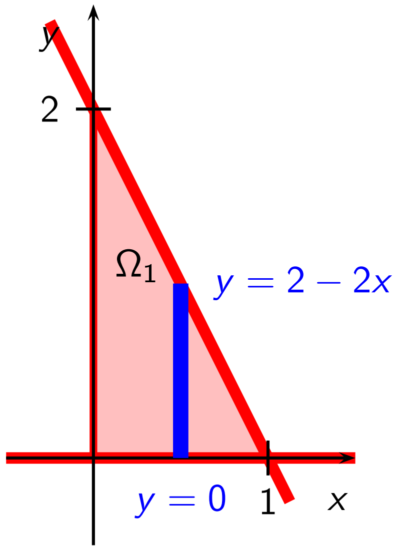
is calculated and reverse the order of integration.

The integration is first with respect to y.

$$I_1 = \int_0^1 \int_0^{2-2x} f(x,y)\; dy\; dx$$

For each value of x, the lower limit of integration is $y = 0$ and the upper limit of integration is $y = 2 - 2x$. The resulting function is integrated for x from 0 to 1.

The region of integration Ω_1 is the triangle with vertices $(0,0)$, $(0,2)$ and $(1,0)$.

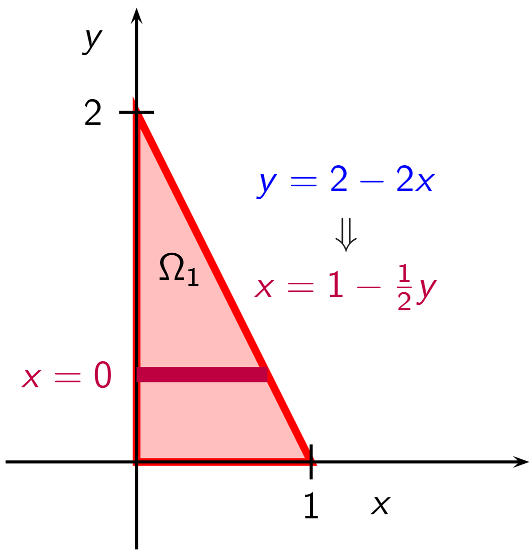


Fubini's theorem examples

To reverse the order of integration, take slices of constant y and integrate first with respect to x.

For each value of y, the lower limit of integration is $x = 0$ and the upper limit of integration is $x = 1 - \frac{1}{2}y$. The resulting function is integrated with respect to y from 0 to 2.

$$I_1 = \int_0^2 \int_0^{1-\frac{1}{2}y} f(x,y)\; dx\; dy.$$



So,

$$I_1 = \int_0^1 \int_0^{2-2x} f(x,y)\; dydx = \int_0^2 \int_0^{1-\frac{1}{2}y} f(x,y)\; dxdy.$$

Fubini's theorem examples

Example (b) (ii): Find the region over which the integral

$$I_2 = \int_{-1}^{1} \int_{x-1}^{\sqrt{1-x^2}} f(x,y)\ dydx$$

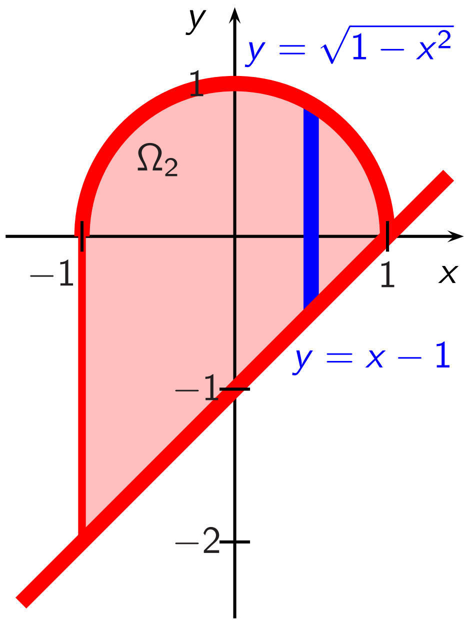
is calculated and reverse the order of integration.

The integration is first with respect to y.

$$I_2 = \int_{-1}^{1} \int_{x-1}^{\sqrt{1-x^2}} f(x,y)\ dy\ dx$$

For each value of x, the lower limit of integration is $y = x - 1$ and the upper limit of integration is $y = \sqrt{1-x^2}$. The resulting function is integrated for x from -1 to 1.

The region of integration Ω_2 is the triangle with vertices $(-1, 0)$, $(0, 1)$ and $(-1, -2)$ with a semi-circular cap on top.

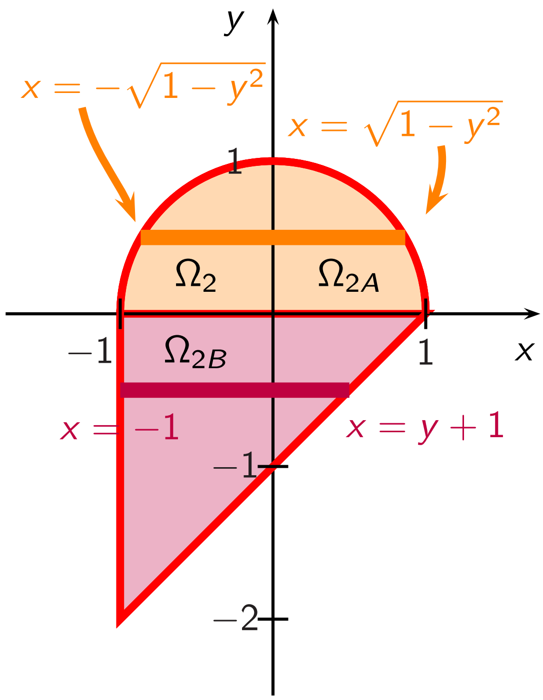


Fubini's theorem examples

To reverse the order of integration and integrate with respect to x first, it is convenient to split the region into two pieces.

$$I_{2A} = \int_{0}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\ dx\ dy.$$

$$I_{2B} = \int_{-2}^{0} \int_{-1}^{y+1} f(x,y)\ dx\ dy.$$



So,

$$I_2 = \int_{-1}^{1} \int_{x-1}^{\sqrt{1-x^2}} f(x,y)\ dydx$$

$$= \int_{0}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\ dxdy + \int_{-2}^{0} \int_{-1}^{y+1} f(x,y)\ dxdy.$$

Uniform continuity

Definition

A function $\mathbf{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is uniformly continuous on Ω if for all $\mathbf{x}, \mathbf{y} \in \Omega$ and for all positive $\epsilon \in \mathbb{R}$ there exists δ such that

$$d(\mathbf{x}, \mathbf{y}) < \delta \ \Rightarrow \ d\Big(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{y})\Big) < \epsilon.$$

In the definition of continuity, δ may depend on $\mathbf{x}$, but here, given ϵ, the same δ must work for all $\mathbf{x}$.

Theorem

A continuous function on a compact set Ω is uniformly continuous on Ω.

Leibniz' rule

Theorem (Leibniz' rule for differentiation under the integral sign)

Consider a function $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ that is continuous on Ω and $\dfrac{\partial f}{\partial x}$ is uniformly continuous on Ω. If

$$F(x) = \int_a^b f(x, y) dy$$

then

$$\begin{aligned} F'(x) &= \frac{d}{dx} \int_a^b f(x, y) dy \\ &= \int_a^b \frac{\partial f}{\partial x}(x, y) dy. \end{aligned}$$

A similar rule applies for improper integrals.

[Given any of Leibniz' rule, Fubini's theorem or Clairot's theorem, the others can be proved. See Q9 on tutorial sheet 4.]

Proof of Leibniz' rule

Given

$$F(x) = \int_a^b f(x,y)dy,$$

we want to show that

$$F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \int_a^b \frac{\partial f}{\partial x}(x,y)dy.$$

So consider

$$\left| \frac{F(x+h)-F(x)}{h} - \int_a^b \frac{\partial f}{\partial x}(x,y)dy \right|$$

$$= \left| \frac{\int_a^b f(x+h,y)dy - \int_a^b f(x,y)dy}{h} - \int_a^b \frac{\partial f}{\partial x}(x,y)dy \right|$$

$$= \left| \int_a^b \frac{f(x+h,y)-f(x,y)}{h} - \frac{\partial f}{\partial x}(x,y)\ dy \right|$$

$$= \left| \int_a^b \frac{\partial f}{\partial x}(c,y) - \frac{\partial f}{\partial x}(x,y)\ dy \right|$$

[for some c between x and $x+h$ by the Mean Value Theorem]

Proof of Leibniz' rule

$$\left| \frac{F(x+h)-F(x)}{h} - \int_a^b \frac{\partial f}{\partial x}(x,y)dy \right| = \left| \int_a^b \frac{\partial f}{\partial x}(c,y) - \frac{\partial f}{\partial x}(x,y)\ dy \right|$$

$$\leq \int_a^b \left| \frac{\partial f}{\partial x}(c,y) - \frac{\partial f}{\partial x}(x,y) \right| dy$$

$$\left| \int_\Omega g \right| \leq \int_\Omega |g|$$

Since $\frac{\partial f}{\partial x}$ is uniformly continuous, for any $\epsilon' > 0$ there is a δ (independent of c, x and y) such that

$$\left\| (c,y)-(x,y) \right\| < \delta \ \Rightarrow \left| \frac{\partial f}{\partial x}(c,y) - \frac{\partial f}{\partial x}(x,y) \right| < \epsilon'$$

$$\Rightarrow \int_a^b \left| \frac{\partial f}{\partial x}(c,y) - \frac{\partial f}{\partial x}(x,y) \right| dy < \epsilon'(b-a)$$

Proof of Leibniz' rule

So for any $\epsilon > 0$ $\left(\text{choose } \epsilon' = \dfrac{\epsilon}{b-a}\right)$ we can choose δ so that

$$\begin{aligned}
|h| < \delta \;\Rightarrow\; |c - x| < \delta \;\Rightarrow\; \Big|\Big|(c,y) - (x,y)\Big|\Big| < \delta \\
\Rightarrow \int_a^b \left|\frac{\partial f}{\partial x}(c,y) - \frac{\partial f}{\partial x}(x,y)\right| dy < \epsilon \\
\Rightarrow \left|\frac{F(x+h) - F(x)}{h} - \int_a^b \frac{\partial f}{\partial x}(x,y)dy\right| < \epsilon.
\end{aligned}$$

So,

$$\lim_{h \to 0}\left|\frac{F(x+h) - F(x)}{h} - \int_a^b \frac{\partial f}{\partial x}(x,y)dy\right| = 0,$$

that is,

$$F'(x) = \lim_{h \to 0}\frac{F(x+h) - F(x)}{h} = \int_a^b \frac{\partial f}{\partial x}(x,y)dy.$$

Leibniz' rule examples

Use the fact that for $a, b > 0$,

$$\int_0^1 \frac{dx}{\sqrt{ax+b}} = \frac{2}{a}\left(\sqrt{a+b} - \sqrt{b}\right)$$

to find

(a) $\displaystyle\int_0^1 \frac{xdx}{(ax+b)^{3/2}}$

(b) $\displaystyle\int_0^1 \frac{dx}{(ax+b)^{3/2}}$

Leibniz' rule examples

(a) $\Omega = [c,d] \times [0,1]$ for $0 < c < d$ is compact and $f : \Omega \to \mathbb{R}$ with

$$f(a,x) = \frac{1}{\sqrt{ax+b}} \quad \text{and} \quad \frac{\partial f}{\partial a}(a,x) = -\frac{x}{2(ax+b)^{3/2}} \quad \text{for } b > 0$$

are bounded and continuous and hence are both uniformly continuous on Ω.

$$\frac{\partial}{\partial a}\left(\int_0^1 \frac{1}{\sqrt{ax+b}}dx\right) = \frac{\partial}{\partial a}\left(\frac{2}{a}\left(\sqrt{a+b}-\sqrt{b}\right)\right)$$

$$\Rightarrow \int_0^1 \frac{\partial}{\partial a}\left(\frac{1}{\sqrt{ax+b}}\right)dx = -\frac{2}{a^2}\left(\sqrt{a+b}-\sqrt{b}\right) + \frac{1}{a\sqrt{a+b}}$$

$$\Rightarrow \int_0^1 -\frac{1}{2}\left(\frac{x}{(ax+b)^{3/2}}\right)dx = -\frac{2}{a^2}\left(\sqrt{a+b}-\sqrt{b}\right) + \frac{1}{a\sqrt{a+b}}$$

$$\Rightarrow \int_0^1 \frac{x}{(ax+b)^{3/2}}dx = \frac{4}{a^2}\left(\sqrt{a+b}-\sqrt{b}\right) - \frac{2}{a\sqrt{a+b}}.$$

Leibniz' rule examples

(b) $\Omega = [c,d] \times [0,1]$ for $0 < c < d$ is compact and $f : \Omega \to \mathbb{R}$ with

$$f(b,x) = \frac{1}{\sqrt{ax+b}} \quad \text{and} \quad \frac{\partial f}{\partial b}(b,x) = -\frac{1}{2(ax+b)^{3/2}} \quad \text{for } a > 0$$

are bounded and continuous and hence are both uniformly continuous on Ω.

$$\frac{\partial}{\partial b}\left(\int_0^1 \frac{1}{\sqrt{ax+b}}dx\right) = \frac{\partial}{\partial b}\left(\frac{2}{a}\left(\sqrt{a+b}-\sqrt{b}\right)\right)$$

$$\Rightarrow \int_0^1 \frac{\partial}{\partial b}\left(\frac{1}{\sqrt{ax+b}}\right)dx = \frac{1}{a}\left(\frac{1}{\sqrt{a+b}} - \frac{1}{\sqrt{b}}\right)$$

$$\Rightarrow \int_0^1 -\frac{1}{2}\left(\frac{1}{(ax+b)^{3/2}}\right)dx = \frac{1}{a}\left(\frac{1}{\sqrt{a+b}} - \frac{1}{\sqrt{b}}\right)$$

$$\Rightarrow \int_0^1 \frac{1}{(ax+b)^{3/2}}dx = -\frac{2}{a}\left(\frac{1}{\sqrt{a+b}} - \frac{1}{\sqrt{b}}\right).$$

Leibniz' rule examples

Given that for $a > 0$,

$$\int_0^\infty e^{-ax}\sin(kx)\;dx = \frac{k}{a^2+k^2}$$

find (a) $\displaystyle\int_0^\infty xe^{-ax}\sin(kx)\;dx$ (b) $\displaystyle\int_0^\infty xe^{-ax}\cos(kx)\;dx$.

(a) Assuming suitable continuity and uniform continuity of the integrand and its partial derivative with respect to a,

$$\begin{aligned}
\frac{\partial}{\partial a}\left(\int_0^\infty e^{-ax}\sin(kx)\;dx\right) &= \frac{\partial}{\partial a}\left(\frac{k}{a^2+k^2}\right)\\
\Rightarrow\quad \int_0^\infty \frac{\partial}{\partial a}\left(e^{-ax}\sin(kx)\;dx\right) &= -\frac{2ak}{(a^2+k^2)^2}\\
\Rightarrow\quad \int_0^\infty -xe^{-ax}\sin(kx)dx &= -\frac{2ak}{(a^2+k^2)^2}\\
\Rightarrow\quad \int_0^\infty xe^{-ax}\sin(kx)dx &= \frac{2ak}{(a^2+k^2)^2}.
\end{aligned}$$

Leibniz' rule examples

$$\int_0^\infty e^{-ax}\sin(kx)\;dx = \frac{k}{a^2+k^2}$$

(b) Assuming suitable continuity and uniform continuity of the integrand and its partial derivative with respect to k,

$$\begin{aligned}
\frac{\partial}{\partial k}\left(\int_0^\infty e^{-ax}\sin(kx)\;dx\right) &= \frac{\partial}{\partial k}\left(\frac{k}{a^2+k^2}\right)\\
\Rightarrow\quad \int_0^\infty \frac{\partial}{\partial k}\left(e^{-ax}\sin(kx)\;dx\right) &= \frac{1}{a^2+k^2}-\frac{2k^2}{(a^2+k^2)^2}\\
\Rightarrow\quad \int_0^\infty xe^{-ax}\cos(kx)dx &= \frac{a^2-k^2}{(a^2+k^2)^2}.
\end{aligned}$$

Leibniz' rule with variable limits

To find

$$\frac{d}{dt}\int_{u(t)}^{v(t)} f(x,t)dx$$

we need to use the Fundamental Theorem of Calculus

$$\frac{d}{dt}\int_{a}^{t} f(x,t)dx.$$

Now let $w(t)=t$ and

$$F(u,v,w)=\int_u^v f(x,w)dx.$$

The chain rule says

$$\begin{aligned}
\frac{d}{dt}F(u,v,w) &= \frac{\partial F}{\partial u}\frac{du}{dt}+\frac{\partial F}{\partial v}\frac{dv}{dt}+\frac{\partial F}{\partial w}\cancelto{1}{\frac{dw}{dt}}\\
&= -f(u,w)\frac{du}{dt}+f(v,w)\frac{dv}{dt}+\frac{\partial}{\partial w}\int_u^v f(x,w)dx\\
&= -f(u,t)\frac{du}{dt}+f(v,t)\frac{dv}{dt}+\int_u^v \frac{\partial f}{\partial t}(x,t)dx
\end{aligned}$$

Leibniz' rule with variable limits

Example:

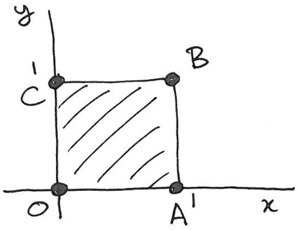
$$\begin{aligned}
\frac{d}{dt}\int_{\sqrt{t}}^{t}\frac{\cos(tx)}{x}dx &= -\frac{\cos(t\sqrt{t})}{\sqrt{t}}\frac{d}{dt}\left(\sqrt{t}\right)+\frac{\cos(t.t)}{t}\frac{dt}{dt}+\int_{\sqrt{t}}^{t}\frac{\partial}{\partial t}\left(\frac{\cos(tx)}{x}\right)dx\\
&= -\frac{\cos(t\sqrt{t})}{\sqrt{t}}\frac{1}{2\sqrt{t}}+\frac{\cos(t^2)}{t}+\int_{\sqrt{t}}^{t}\frac{-x\sin(tx)}{x}dx\\
&= -\frac{\cos(t\sqrt{t})}{2t}+\frac{\cos(t^2)}{t}+\left[\frac{\cos(tx)}{t}\right]_{\sqrt{t}}^{t}\\
&= -\frac{\cos(t\sqrt{t})}{2t}+\frac{\cos(t^2)}{t}+\frac{\cos(t^2)}{t}-\frac{\cos(t\sqrt{t})}{t}\\
&= -\frac{3\cos(t\sqrt{t})}{2t}+\frac{2\cos t^2}{t}
\end{aligned}$$

Change of variables

Change of coordinates

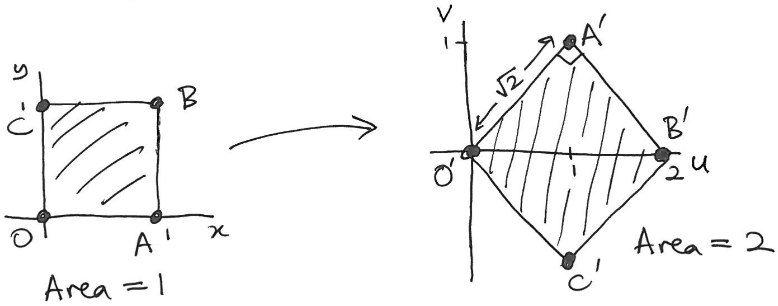
How does the area of OABC change under the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad ?$$



$O: \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad$ $A: \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$B: \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $\qquad$ $C: \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



Area = 1

$$\left|\det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right| = |-1-1| = |-2| = 2$$

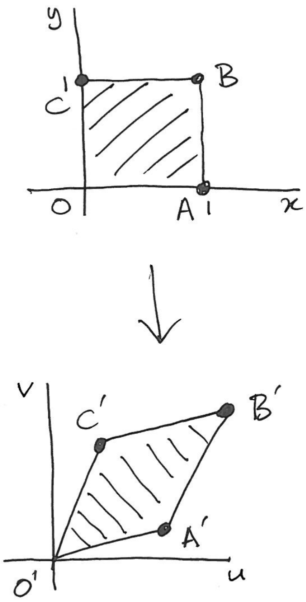
Change of variables

How does area of OABC change under the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$O: \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad$ $A: \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$

$B: \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix}$ $\qquad$ $C: \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$



$\overrightarrow{O'A'} = \overrightarrow{C'B'} = \begin{pmatrix} a \\ c \end{pmatrix}$

so $O'A'B'C'$ is a parallelogram.

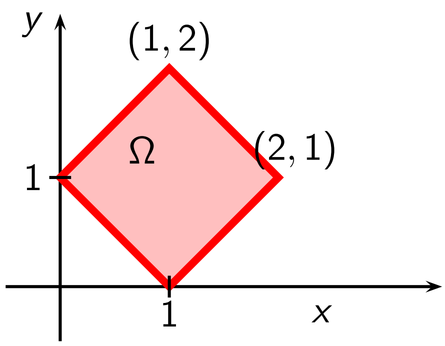
vectors in 3D

$$\text{Area of } O'A'B'C' = \|\overrightarrow{O'A'} \times \overrightarrow{O'C'}\| = \left\| \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a & c & 0 \\ b & d & 0 \end{vmatrix} \right\| = \|\underset{\sim}{i}.0 - \underset{\sim}{j}.0 + \underset{\sim}{k}.(ad-bc)\| = |ad-bc| = \left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|$$

Change of variables

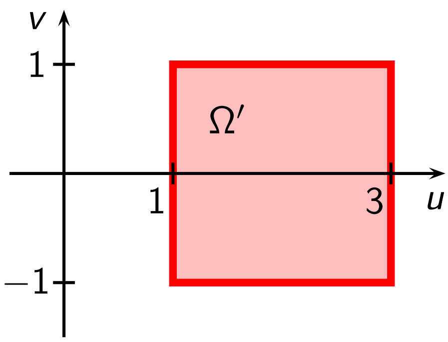
Consider the integral of x over the region Ω shown on the right,

$$I = \int_\Omega x = \int_0^1 \int_{1-x}^{1+x} x \, dydx + \int_1^2 \int_{x-1}^{3-x} x \, dydx.$$



Note that Ω is bounded by the lines

$$x + y = 1, \qquad x + y = 3,$$
$$x - y = 1, \qquad x - y = -1.$$



Perhaps it might be easier to use

$$u = x + y \quad \text{and} \quad v = x - y$$

as coordinates. In the uv-plane, the corresponding region Ω' has boundaries

$$u = 1, \; u = 3, \; v = -1 \text{ and } v = 1.$$

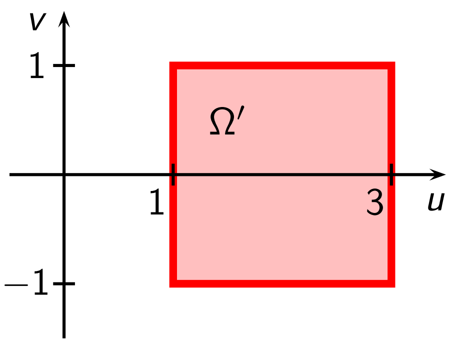
Change of variables

If

$$u = x + y \quad \text{and} \quad v = x - y$$

then

$$x = \frac{1}{2}(u + v).$$



Can we simply write the integral as

$$I = \int_{-1}^1 \int_1^3 \frac{1}{2}(u + v) \; dudv?$$

No! The map

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

scales areas by a factor of $\left| \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right| = 2.$

$$dudv = 2dxdy \;\Rightarrow\; dxdy = \frac{1}{2}dudv.$$

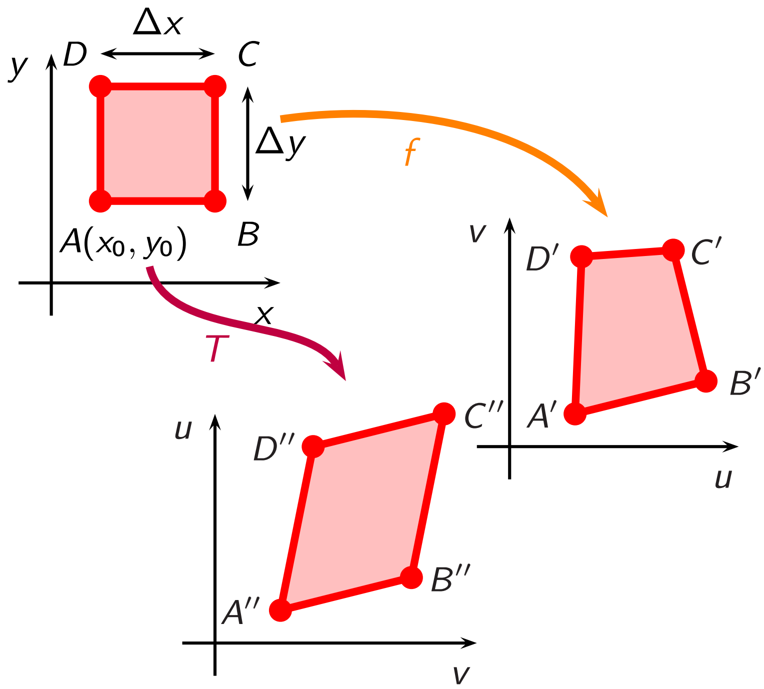
So,

$$I = \int_{-1}^1 \int_1^3 \frac{1}{2}(u + v) \; \frac{1}{2} \; dudv.$$

Change of variables

For a more general change of variables by a differentiable function $f : \mathbb{R} \to \mathbb{R}$

$$\begin{pmatrix} u \\ v \end{pmatrix} = f\begin{pmatrix} x \\ y \end{pmatrix} \simeq T\begin{pmatrix} x \\ y \end{pmatrix} = f\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + Jf\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$



$$A'' \simeq f\binom{x_0}{y_0} + Jf\binom{x_0}{y_0}\binom{0}{0}$$

$$B'' \simeq f\binom{x_0}{y_0} + Jf\binom{x_0}{y_0}\binom{\Delta x}{0}$$

$$C'' \simeq f\binom{x_0}{y_0} + Jf\binom{x_0}{y_0}\binom{\Delta x}{\Delta y}$$

$$D'' \simeq f\binom{x_0}{y_0} + Jf\binom{x_0}{y_0}\binom{0}{\Delta y}$$

Area $A'B'C'D'$

$\simeq$ Area $A''B''C''D''$

$=$ Area $ABCD \quad \times \quad |\mathrm{det} JF|$.

Change of variables

Theorem (Change of variables)

Suppose $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is C^1, $det(J_\mathbf{x} F) \neq 0$ for $\mathbf{x} \in \Omega$ and F is one-to-one. Then, if f is integrable on $\Omega' = F(\Omega)$

$$\int_{\Omega'} f = \int_{\Omega} (f \circ F)\,|detJF|$$

Alternative notation:

$$\int_{\Omega'} f(x,y)\; dxdy = \int_{\Omega} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \; dudv$$

where

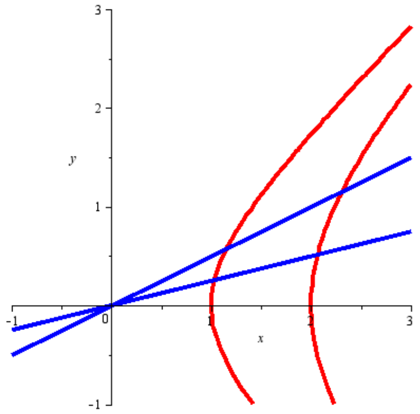
$$\frac{\partial(x,y)}{\partial(u,v)} = \det(JF) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Change of variables

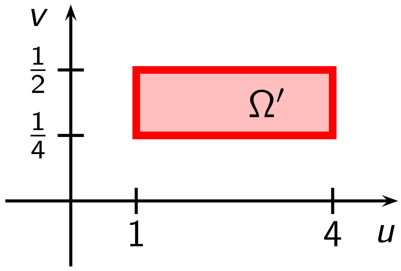
Find the area of the region Ω bounded by

$$x^2 - y^2 = 1, \qquad x^2 - y^2 = 4$$

$$y = \frac{x}{2}, \qquad y = \frac{x}{4}.$$



Let $u = x^2 - y^2$ and $v = \dfrac{y}{x}$.



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\text{Area} = \int_\Omega 1 = \iint_\Omega 1 \, dxdy = \iint_{\Omega'} 1 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv.$$

Change of variables

$$\begin{pmatrix} x \\ y \end{pmatrix} = F \begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow F^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ \frac{y}{x} \end{pmatrix}$$

Note that F^{-1} is differentiable for $x \neq 0$. So

$$J(F^{-1}) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & -2y \\ -\frac{y}{x^2} & \frac{1}{x} \end{pmatrix}.$$

$$\det\left(J(F^{-1})\right) = 2x\,\frac{1}{x} - (-2y)\left(-\frac{y}{x^2}\right) = 2 - 2\,\frac{y^2}{x^2} = 2 - 2v^2.$$

But we want

$$\det(JF) = \frac{1}{\det\left((JF)^{-1}\right)} = \frac{1}{\det\left(J(F^{-1})\right)} = \frac{1}{2 - 2v^2},$$

that is,

$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = \frac{1}{2 - 2v^2}.$$

Change of variables

$$\text{Area} = \iint_{R'} 1 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$= \int_1^4 \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{2-2v^2}\, dv\,du$$

note: $\frac{1}{2-2v^2} > 0$ for $\frac{1}{4} \le v \le \frac{1}{2}$

$$= \int_1^4 du \cdot \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{2-2v^2}\, dv$$

$$= \int_1^4 du \quad \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{4}\left(\frac{1}{1+v} + \frac{1}{1-v}\right) dv$$

$$= \quad 3 \quad \cdot \frac{1}{4}\Big[\ln(1+v) - \ln(1-v)\Big]_{\frac{1}{4}}^{\frac{1}{2}}$$

$$= \frac{3}{4}\left(\ln(\tfrac{3}{2}) - \ln(\tfrac{1}{2}) - \ln(\tfrac{5}{4}) + \ln(\tfrac{3}{4})\right)$$

$$= \frac{3}{4}(\ln 3 - \ln 2 + \ln 2 - \ln 5 + \ln 4 + \ln 3 - \ln 4)$$

$$= \frac{3}{4}(2\ln 3 - \ln 5)$$

Change of variables

Integrate x over the part of the unit disc that lies in the first quadrant.

$$\begin{pmatrix} x \\ y \end{pmatrix} = F\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

$$\therefore \quad JF = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\therefore \det(JF) = r\cos^2\theta + r\sin^2\theta = r$$

So $\left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = |\det(JF)| = |r| = r$

(note $\det(JF) = 0$ where $r = 0$!)

Change of variables

$$\therefore \iint_R x\,dx\,dy = \iint_{R'} r\cos\theta \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 r\cos\theta \; r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 r^2\cos\theta \, dr\,d\theta$$

$$= \int_0^1 r^2\,dr \cdot \int_0^{\frac{\pi}{2}} \cos\theta\,d\theta$$

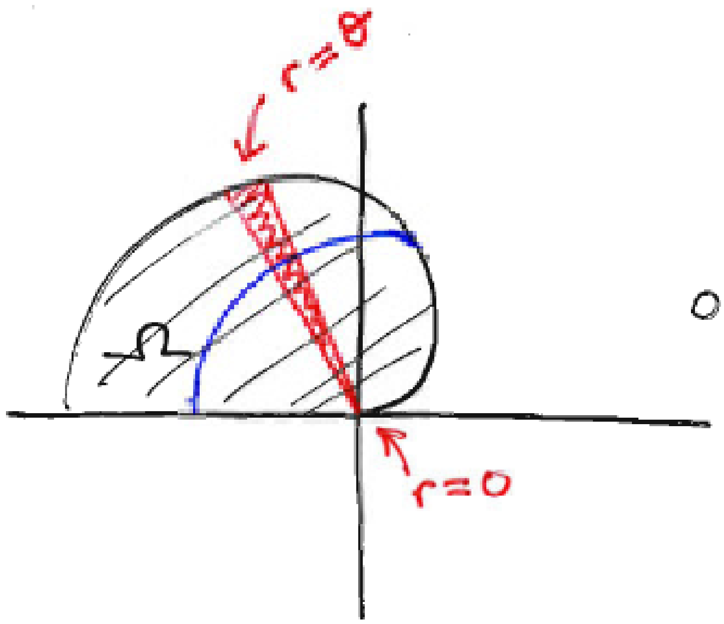
$$= \left[\tfrac{1}{3}r^3\right]_0^1 \left[\sin\theta\right]_0^{\pi/2}$$

$$= \left(\tfrac{1}{3}-0\right)\left(1-0\right)$$

$$= \tfrac{1}{3}$$

Change of variables

Find the area of the region bounded by the spiral $r=\theta$ and the x-axis.



$0 \leq \theta \leq \pi$

Normal distribution

The standard normal distribution $p(x) = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$ is a probability distribution on $\mathbb{R}$. But how do we know that $\displaystyle\int_{-\infty}^{\infty} p(x)dx = 1$?

If $I = \displaystyle\int_0^{\infty} e^{-x^2}dx$, then

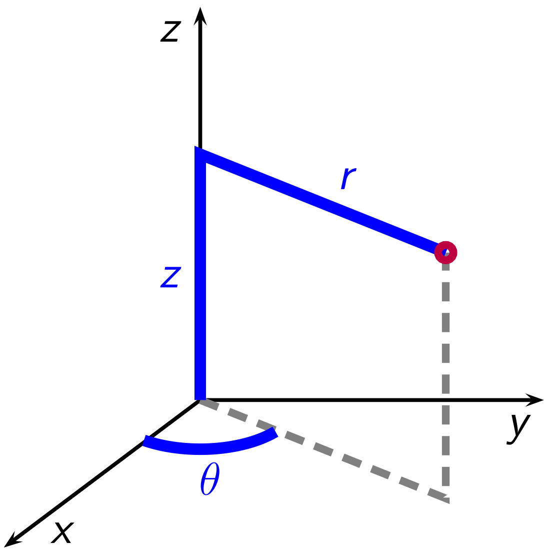
$$
\begin{aligned}
I^2 &= \int_0^{\infty} e^{-x^2}dx \quad \int_0^{\infty} e^{-y^2}dy \\
&= \int_0^{\infty}\int_0^{\infty} e^{-x^2}e^{-y^2}\ dxdy \\
&= \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\ dxdy \\
&= \int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2}\ rdrd\theta \\
&= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\infty} e^{-r^2}\ rdr \\
&= \left[\frac{1}{2}\theta^2\right]_0^{\frac{\pi}{2}} \left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} \\
&= \frac{\pi}{2}\left(0 - \left(-\frac{1}{2}e^0\right)\right) \\
&= \frac{\pi}{4}.
\end{aligned}
$$

So,

$$I = \int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$$

Cylindrical and spherical polar coordinates

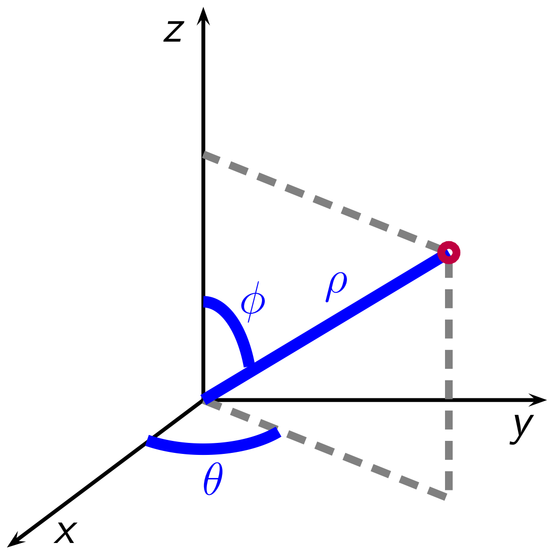
Cylindrical polar coordinates



$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r$$

Spherical polar coordinates



$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi$$

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \rho^2\sin\phi$$

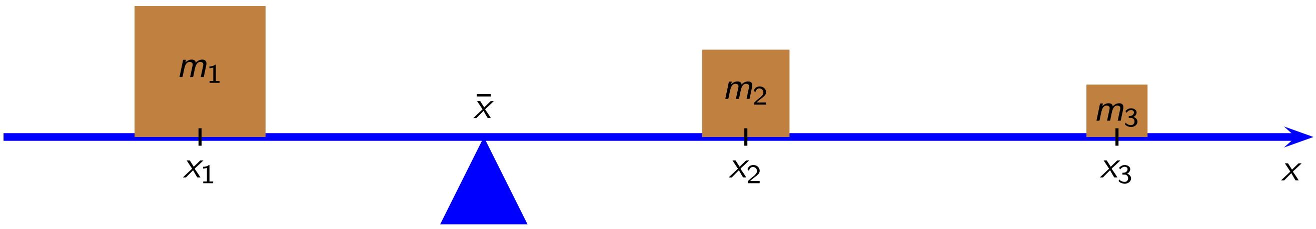
Spherical and polar coordinates

Example (Exam 2008):

Using cylindrical or spherical polar coordinates, write down iterated integrals that would give the volume of the region bounded below by $z = \sqrt{3x^2 + 3y^2}$ and above by $z = \sqrt{4 - x^2 - y^2}$.

Mass, centre of mass, centroid

The *balance point* of masses on a line is the point $\bar{x}$ about which the torque is 0.



$$0 = m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) + m_3(x_3 - \bar{x})$$
$$= \sum_{i=1}^{3} m_i(x_i - \bar{x}) = \sum_{i=1}^{3} m_i x_i - \bar{x}\sum_{i+1}^{3} m_i.$$
$$\Rightarrow \quad \bar{x} = \frac{\sum_{i=1}^{3} m_i x_i}{\sum_{i+1}^{3} m_i}.$$

The total mass is

$$M = \sum_{i+1}^{3} m_i$$

and

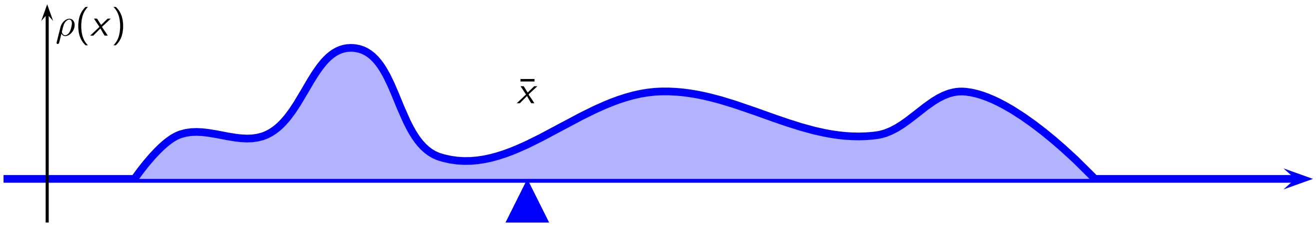
$$p_k = \frac{m_k}{M}$$

is a discrete probability distribution with

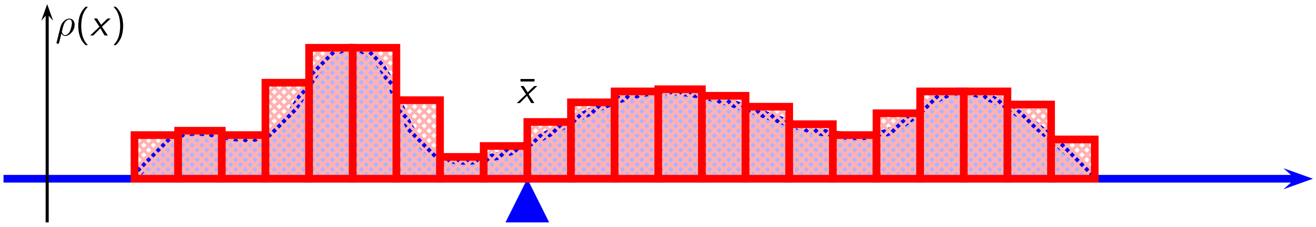
$$\bar{x} = E(X).$$

Mass, centre of mass, centroid

Consider a continuous mass distribution $\rho : \mathbb{R} \to \mathbb{R}$.



We can find the approximate centre of mass using an upper or lower sum of ρ with respect to a partition P.



$$\bar{x} \simeq \frac{\displaystyle\sum_{i=1}^{n} \bar{\rho}_i \Delta x_i \; x_i}{\displaystyle\sum_{i=1}^{n} \bar{\rho}_i \Delta x_i}, \qquad \bar{x} = \frac{\displaystyle\int_a^b \rho(x) \; x \; dx}{\displaystyle\int_a^b \rho(x) \; dx}.$$

$$\left[\begin{array}{l} M = \displaystyle\int_a^b \rho(x) \; dx \text{ is the total mass} \\ \text{and } \dfrac{\rho(x)}{M} \text{ is a probability density.} \end{array} \right]$$

Mass, centre of mass, centroid

For a lamina occupying the region $\Omega \subset \mathbb{R}^2$ with density $\rho : \Omega \to \mathbb{R}$, the total mass is

$$M = \iint_\Omega \rho(x, y) \; dxdy$$

The coordinates of the centre of mass are

$$\bar{x} = \frac{1}{M} \iint_\Omega x\rho(x, y) \; dxdy$$
$$\bar{y} = \frac{1}{M} \iint_\Omega y\rho(x, y) \; dxdy$$

For a solid body occupying the region $\Omega \subset \mathbb{R}^3$ with density $\rho : \Omega \to \mathbb{R}$, the total mass is

$$M = \iiint_\Omega \rho(x, y, z) \; dxdydz$$

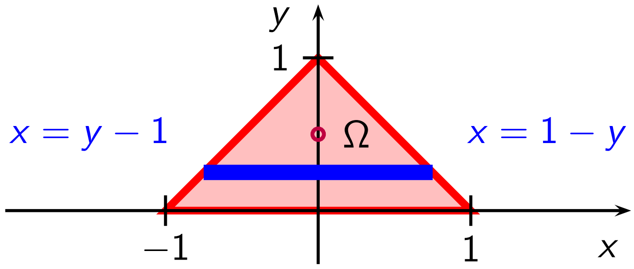
The coordinates of the centre of mass are

$$\bar{x} = \frac{1}{M} \iiint_\Omega x\rho(x, y, z) \; dxdydz$$
$$\bar{y} = \frac{1}{M} \iiint_\Omega y\rho(x, y, z) \; dxdydz$$
$$\bar{z} = \frac{1}{M} \iiint_\Omega z\rho(x, y, z) \; dxdydz$$

For a lamina or solid body of constant density, the centre of mass is called the centroid and denoted (x_c, y_c, z_c).

Mass, centre of mass, centroid

Example: Find the centre of mass of the trianglular lamina with vertices at $(-1, 0)$, $(0, 1)$ and $(1, 0)$ with density $\rho(x, y) = y$.



$$M = \int_\Omega y = \int_0^1 \int_{y-1}^{1-y} y \; dx \; dy$$
$$= \int_0^1 \Big[yx\Big]_{y-1}^{1-y} \; dy$$
$$= \int_0^1 y(1-y) - y(y-1) \; dy$$
$$= \frac{1}{3}.$$

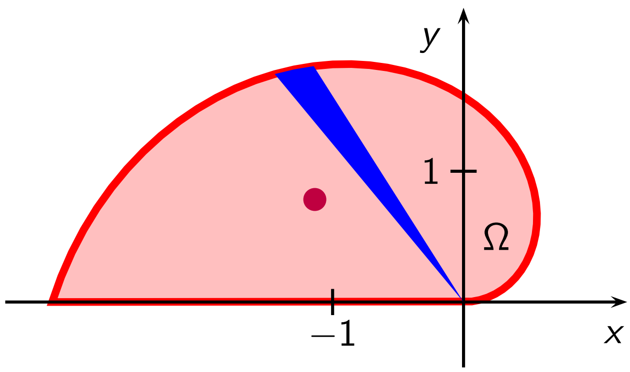
The symmetry of Ω and ρ gives $\bar{x} = 0$.

$$\int_\Omega y\rho = \int_0^1 \int_{y-1}^{1-y} y \; y \; dx \; dy$$
$$= \int_0^1 \Big[y^2 x\Big]_{y-1}^{1-y} \; dy$$
$$= \int_0^1 y^2(1-y) - y^2(y-1) \; dy$$
$$= \frac{1}{6}.$$

$$\bar{y} = \frac{1}{M}\int_\Omega y\rho = \frac{1}{\frac{1}{3}} \, \frac{1}{6} = \frac{1}{2}.$$

Mass, centre of mass, centroid

Example: Find the centroid of the region bounded by $r = \theta$ and the x-axis.



To calculate the centroid, take $\rho = 1$. We have already calculated the total mass (area) to be $A = \pi^3/6$.

$$\int_\Omega x = \int_0^\pi \int_0^\theta r\cos\theta \; rdr \; d\theta$$
$$= \int_0^\pi \left[\frac{1}{3} r^3 \cos\theta\right]_0^\theta \; d\theta$$

$$= \int_0^\pi \frac{1}{3}\theta^3 \cos\theta \; d\theta$$
$$= 4 - \pi^2 \quad \text{(int by parts 3 times)}$$

$$x_c = \frac{6(4-\pi^2)}{\pi^3} \simeq -1.1358.$$

A similar calculation gives

$$\int_\Omega y = \int_0^\pi \int_0^\theta r\sin\theta \; rdr \; d\theta = \frac{\pi^3 - 6\pi}{3}.$$

$$y_c = \frac{2(\pi^2 - 6)}{\pi^2} \simeq 0.78415.$$