MATH2111 Higher Several Variable Calculus
Curves and Surfaces

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Welcome to MATH2111

- **Lecturer weeks 1 to 5 and 12:** Dr Jonathan Kress
- **Lecturer weeks 6 to 11:** A/Prof Josef Dick
- Tutorials start in week 2.
- Moodle `http://moodle.telt.unsw.edu.au` will have lecture notes, videos of the week 1 to 6 lectures from 2011 lectures, tutorial problems etc.
- MATH2111 is a significantly more difficult than MATH2011. If you have any concerns about whether you should take MATH2011 or MATH2111, please discuss this with the lecturer as soon as possible.
- Read the course outline - a link is on the Moodle course page.
- Information on the writing assignment will be available in week 3.

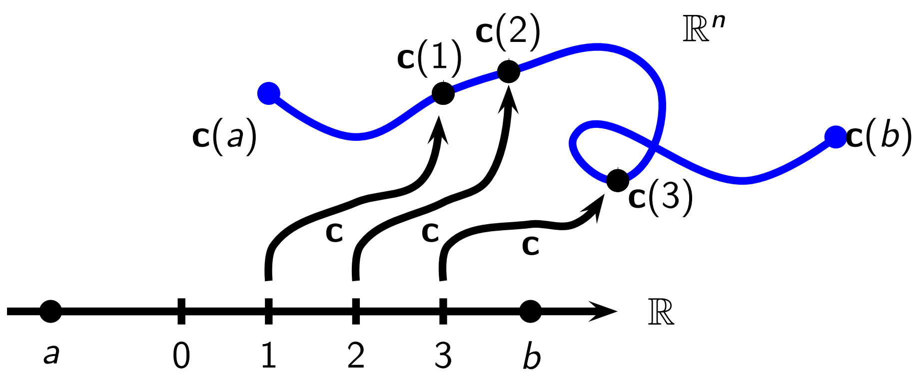
Curves

Definition

A curve in $\mathbb{R}^n$ is a vector valued function

$$\mathbf{c} : I \to \mathbb{R}^n$$

where I is an interval on $\mathbb{R}$.



Often we think of the image of I under $\mathbf{c}$ as the curve, but this is not the definition.

The function $\mathbf{c}$ is also called a parameterisation.

Curves

Example: A curve (or parameterisation) $\mathbf{r} : [-1, 3] \to \mathbb{R}^2$ is given by

$$\mathbf{r}(t) = \left(1 + t, \tfrac{4}{3}t^2\right).$$

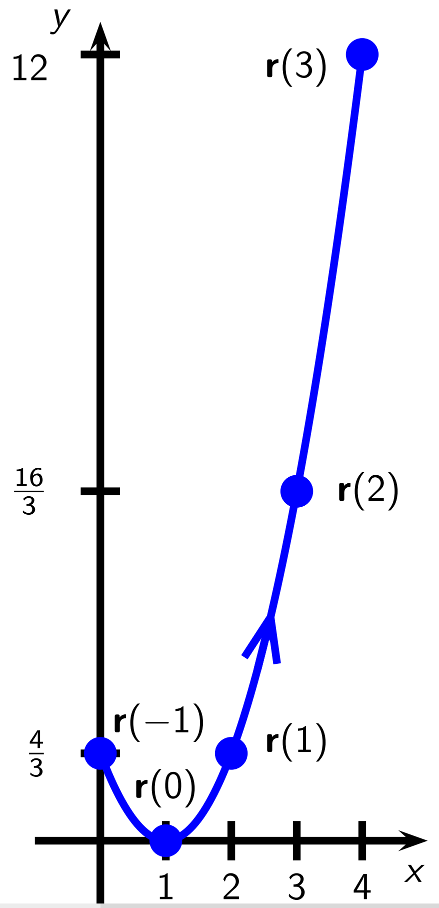
The image of $[-1, 3]$ is

$$\left\{(x, y) : x = 1 + t, y = \tfrac{4}{3}t^2, -1 \le t \le 3\right\}.$$

Plot and label the points $\mathbf{r}(-1)$, $\mathbf{r}(0)$, $\mathbf{r}(1)$, $\mathbf{r}(2)$ and $\mathbf{r}(3)$. Find a Cartesian equation for the image of $[-1, 3]$ and sketch the curve. Indicate the direction of increasing parameter.

A Cartesian equation can be obtained by eliminating the parameter.

$$x = 1 + t, y = \frac{4}{3}t^2 \qquad \Rightarrow \qquad y = \frac{4}{3}(x - 1)^2.$$



Curves

Examples: Sketch the following curves.

$\mathbf{r} : [0, \pi] \to \mathbb{R}^2$ given by

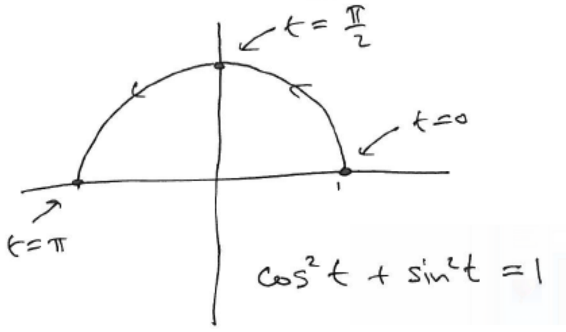
$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}.$$

$\mathbf{r} : [0, 6\pi] \to \mathbb{R}^3$ given by

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}.$$

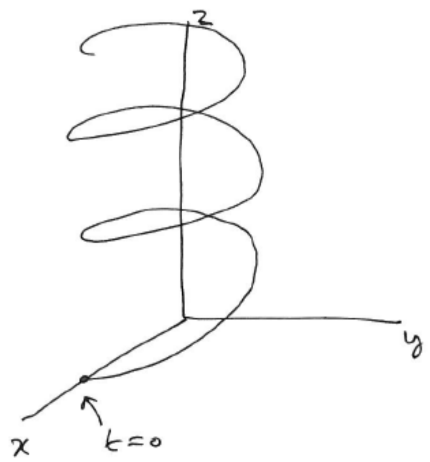
$\underset{\sim}{r} : [0, \pi] \to \mathbb{R}^2$

$\underset{\sim}{r}(t) = \cos t \underset{\sim}{i} + \sin t \underset{\sim}{j}$



$\underset{\sim}{r} : [0, 6\pi] \to \mathbb{R}^3$

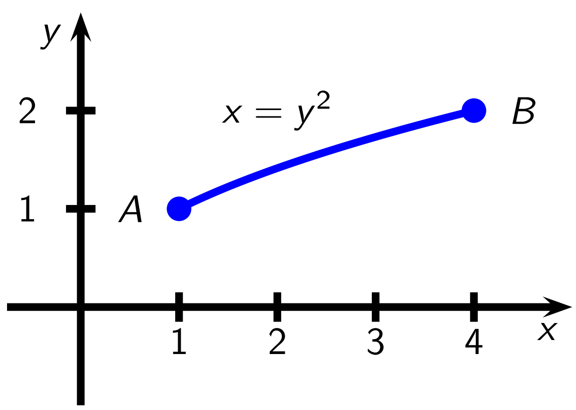
$\underset{\sim}{r}(t) = \cos t \underset{\sim}{i} + \sin t \underset{\sim}{j} + t \underset{\sim}{k}$



Curves

Example

Find two different curves with the image drawn below. For each curve, describe the direction of increasing parameter.



Give a parameterisation that traverses from B to A and another that traverses from A to B and then back to A again.

Each value of x corresponds to only one point in the image. So we can use x as a parameter.

$$\mathbf{r}_1 : [1, 4] \to \mathbb{R}^2,$$

$$\mathbf{r}_1(t) = (t, \sqrt{t}).$$

The parameter increases from A to B.

We could also use y as a parameter.

$$\mathbf{r}_2 : [1, 2] \to \mathbb{R}^2,$$

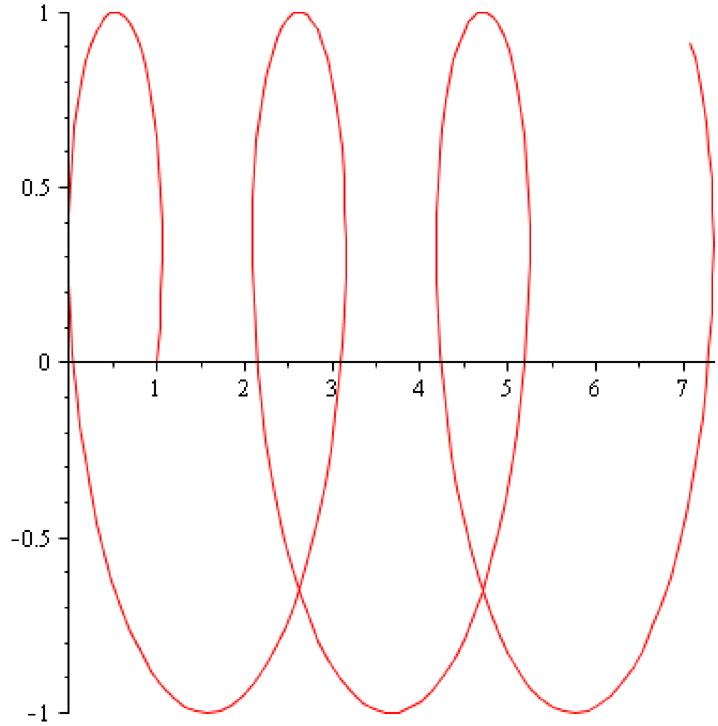
$$\mathbf{r}_2(t) = (t^2, t).$$

The parameter increases from A to B.

Curves

Example: Sketch $\mathbf{r} : [0, 20] \to \mathbb{R}^2$ given by $\mathbf{r}(t) = \left(\cos t + \frac{1}{3}t\right)\mathbf{i} + \sin t\mathbf{j}$.

```
> plot([cos(t)+t/3,sin(t),t=0..20]);
```

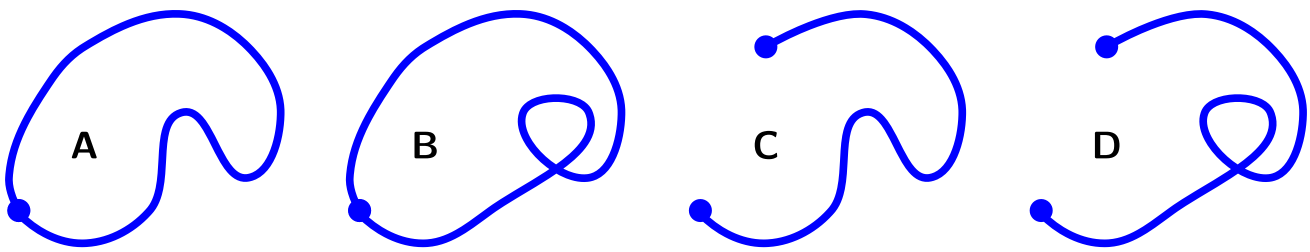


Curves

Definition

- A multiple point is a point through which the curve passes more than once.
- For a curve $\mathbf{c} : [a, b] \to \mathbb{R}^n$, $\mathbf{c}(a)$ and $\mathbf{c}(b)$ are called end points.
- A curve is closed if its end points are the same point.

Which of the following are the image of a closed curve? How many multiple points (other than end points) does each curve have?



A and **B** are closed. **B** and **D** have one multiple point each.

What assumption has been made in the above answers?

Limits and Calculus for Curves

Definition

For an interval $I \subset \mathbb{R}$ and curve $\mathbf{c} : I \to \mathbb{R}^n$ with

$$\mathbf{c}(t) = \Big(c_1(t), c_2(t), \dots, c_n(t)\Big),$$

the functions $c_i : I \to \mathbb{R}$, $i = 1, 2, \dots, n$ are called the components of $\mathbf{c}$.

Define limits, derivatives and integrals component by component.

Definition

- $\lim\limits_{t \to a} \mathbf{c}(t) = \Big(\lim\limits_{t \to a} c_1(t), \lim\limits_{t \to a} c_2(t), \dots, \lim\limits_{t \to a} c_n(t)\Big)$
- $\dfrac{d\mathbf{c}(t)}{dt} = \dot{\mathbf{c}}(t) = \mathbf{c}'(t) = (c_1'(t), c_2'(t), \dots, c_n'(t))$
- $\displaystyle\int_a^b \mathbf{c}(t)dt = \left(\int_a^b c_1(t)dt, \int_a^b c_2(t)dt, \dots, \int_a^b c_n(t)dt\right)$

Limits and Calculus for Curves

Definition

A curve $\mathbf{c} : I \to R^n$ is

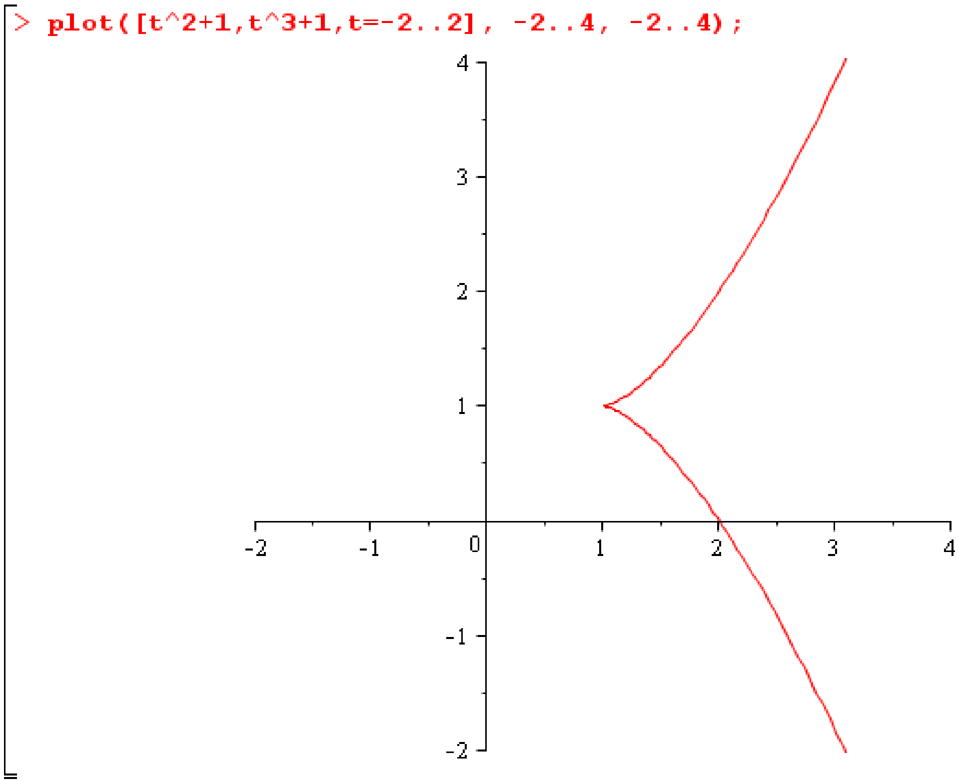
- continuous if its component functions are continuous.
- simple if it is continuous and has no multiple points (other than the end points if it is closed).
- smooth if its components are differentiable and their derivatives do not simultaneouly vanish.
- piecewise smooth if it is made up of a finite number of smooth curves.

A curve has an orientation — the direction of increasing t.

We will revisit continuity in the analysis section and give a different definition which we will show is equivalent.

Limits and Calculus for Curves

Example: $\mathbf{r} : [-2, 2] \to \mathbb{R}^2$ with $\mathbf{r}(t) = (t^2 + 1, t^3 + 1)$ is not smooth.



$\mathbf{r}'(t) = (2t, 3t^2)$.

$\Downarrow$

$\mathbf{r}'(0) = (0, 0)$

Hence $\mathbf{r}$ is not smooth.

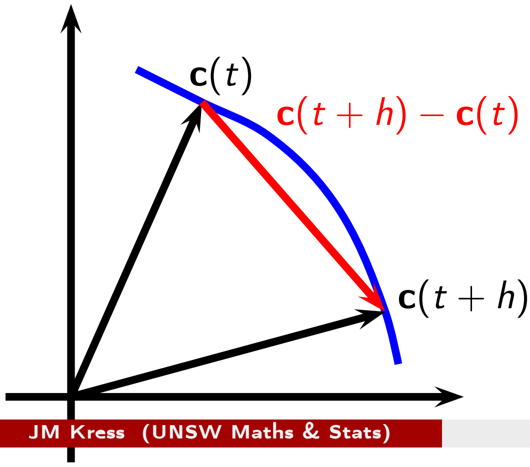
Differentiation Rules for Curves

Working component by component we can prove the following rules from their one variable counterparts.

$$\begin{aligned}
\frac{d}{dt}\Big(\mathbf{c}_1(t) + \mathbf{c}_2(t)\Big) &= \frac{d\mathbf{c}_1(t)}{dt} + \frac{d\mathbf{c}_2(t)}{dt} \\
\frac{d}{dt}\Big(\lambda\mathbf{c}(t)\Big) &= \lambda\frac{d\mathbf{c}(t)}{dt} \\
\frac{d}{dt}\Big(f(t)\mathbf{c}(t)\Big) &= \frac{df(t)}{dt}\mathbf{c}(t) + f(t)\frac{d\mathbf{c}(t)}{dt} \\
\frac{d}{dt}\Big(\mathbf{c}_1(t)\cdot\mathbf{c}_2(t)\Big) &= \frac{d\mathbf{c}_1(t)}{dt}\cdot\mathbf{c}_2(t) + \mathbf{c}_1(t)\cdot\frac{d\mathbf{c}_2(t)}{dt} \\
\frac{d}{dt}\Big(\mathbf{c}_1(t)\times\mathbf{c}_2(t)\Big) &= \frac{d\mathbf{c}_1(t)}{dt}\times\mathbf{c}_2(t) + \mathbf{c}_1(t)\times\frac{d\mathbf{c}_2(t)}{dt} \\
\frac{d}{dt}\Big(\mathbf{c}(f(t))\Big) &= \mathbf{c}'(f(t))\ f'(t)
\end{aligned}$$

Interpretation of the Derivative

$$\begin{aligned}
\frac{d\mathbf{c}(t)}{dt} &= (c_1'(t), \ldots, c_n'(t)) \\
&= \left(\lim_{h\to 0}\frac{c_1(t+h)-c_1(t)}{h}, \ldots, \lim_{h\to 0}\frac{c_n(t+h)-c_n(t)}{h}\right) \\
&= \lim_{h\to 0}\left(\frac{c_1(t+h)-c_1(t)}{h}, \ldots, \frac{c_n(t+h)-c_n(t)}{h}\right) \\
&= \lim_{h\to 0}\frac{\mathbf{c}(t+h)-\mathbf{c}(t)}{h}
\end{aligned}$$



As h gets smaller, the direction of $\mathbf{c}(t+h)-\mathbf{c}(t)$ approaches the direction of the tangent to the curve's image. If $\mathbf{c}'(t)$ exists and is non-zero, it is called the tangent vector to $\mathbf{c}$ at t, or the velocity of $\mathbf{c}$ at t. Ie, $\mathbf{v}(t) = \mathbf{c}'(t)$. The speed of $\mathbf{c}$ at t is $|\mathbf{v}(t)| = \sqrt{\mathbf{v}(t)\cdot\mathbf{v}(t)}$.
The second derivative $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{c}''(t)$ is called the acceleration.

Tangent Vector Example

Consider the curve $\mathbf{r} : I \to \mathbb{R}^3$ for an interval $I \subset \mathbb{R}$ given by

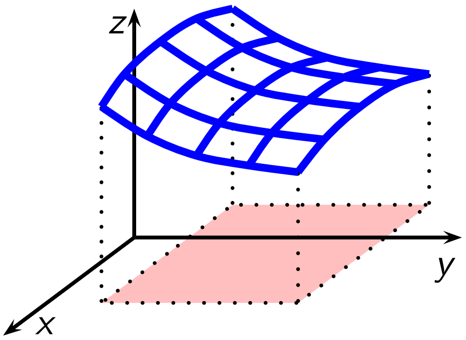
$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 3\sin t\mathbf{j} + \frac{\sqrt{5}}{2}\cos 2t\mathbf{k}.$$

a) Find the velocity and acceleration vectors.
b) Show that the velocity and acceleration vectors are perpendicular at $t = \frac{n\pi}{2}$, $n \in \mathbb{Z}$.
c) Find the length of the curve between $\mathbf{r}(0)$ and $\mathbf{r}(2\pi)$.
[Recall: length $= \int_a^b ||\mathbf{r}'(t)||dt$.]
d) Find the unit tangent vector at $t = \dfrac{\pi}{6}$.
e) Sketch the curve and indicate the unit tangent vector found in (d).

Surfaces

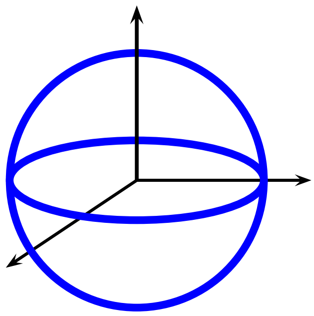
You have seen surfaces in $\mathbb{R}^3$ described in 3 ways.

Graph of a function



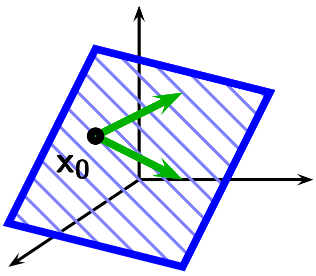
Eg, $z = f(x, y)$

Implicitly



Eg, a sphere given by $x^2 + y^2 + z^2 = 1$.

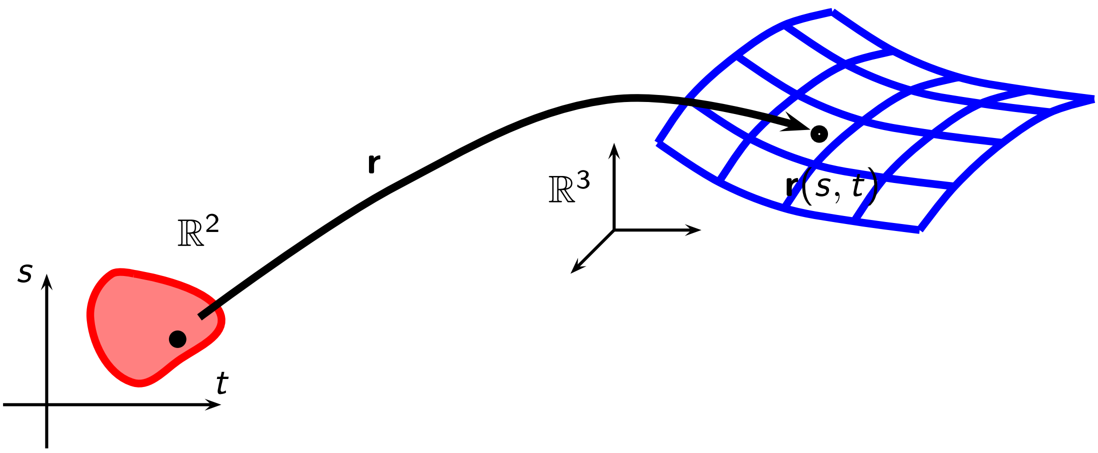
Parametrically



Eg, a plane given by $\mathbf{x} = \mathbf{x_0} + \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2$.

Parameterisation defined surface

For $D \subset \mathbb{R}^2$, the image of D under $\mathbf{r} : D \to \mathbb{R}^3$ is a surface in $\mathbb{R}^3$. Note that unlike for curves, a surface is the image of the parameterisation.



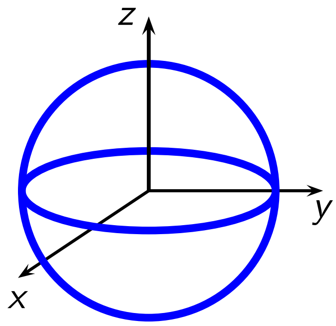
Eg, $\mathbf{r} : D \to \mathbb{R}^3$ where $D = \{(x, y) : x^2 + y^2 \le 1\}$ and

$$\mathbf{r}(s, t) = (s, t, \sqrt{1 - s^2 - t^2})$$

is a parameterisation of the upper unit hemisphere.

Implicitly defined surface

We can define a surface in $\mathbb{R}^3$ as the set of points satisfying an equation. Eg, a sphere given by $x^2 + y^2 + z^2 = 1$.



Later in the course we will study a theorem that tells you when parts of this surface are the graph of a function of some of the variables — the Implicit Function Theorem.

Some other implicitly defined surfaces will be discussed in tutorial 1.

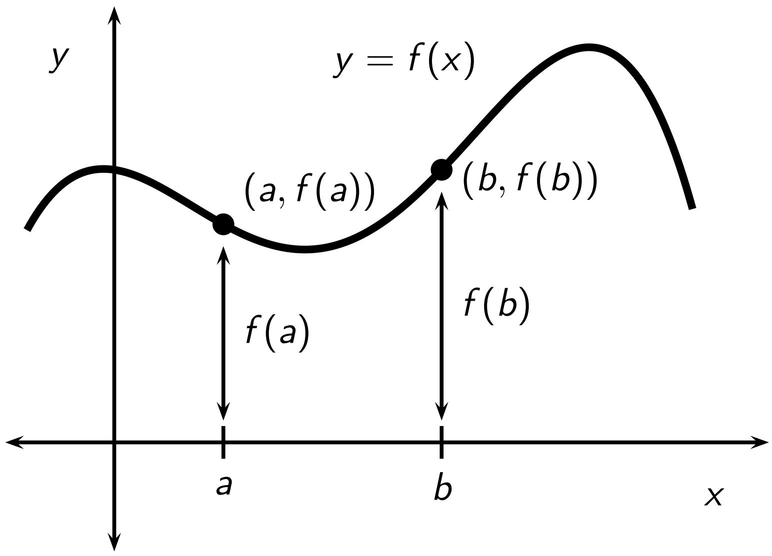
Graphs of functions of one variable

The graph of

$$f : \mathbb{R} \to \mathbb{R}$$

is the set of points

$$\{(x, y) \in \mathbb{R}^2 : y = f(x)\}.$$



On the graph of f, input values are represented by distance across the page and output values by distance up the page.

Graphs of functions of two variables

The graph of

$$f : D \to \mathbb{R}$$

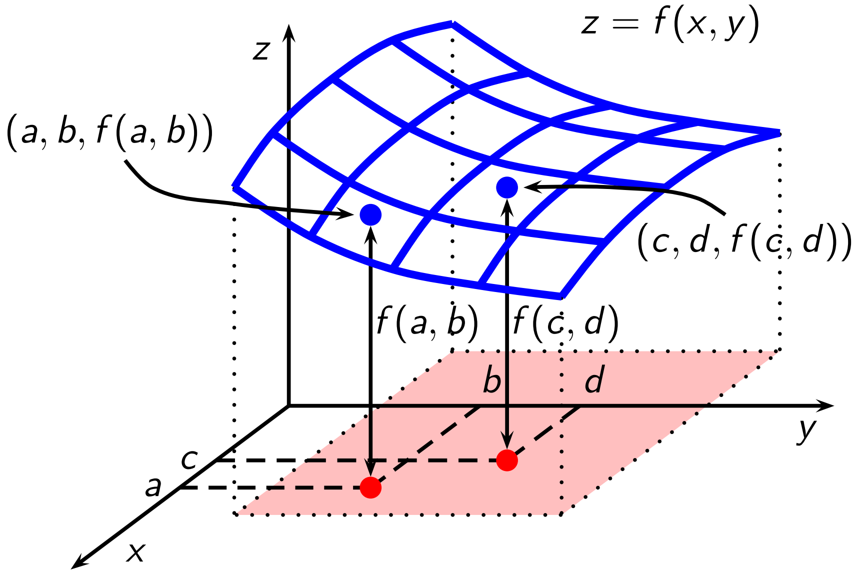
is the set of points

$$\{(x, y, z) : z = f(x, y)$$

$$\text{for all } (x, y) \in D\}.$$

In this example the domain is the subset of $\mathbb{R}^2$ shaded pink in the diagram.

In other examples, it could be all of $\mathbb{R}^2$ or any other subset of $\mathbb{R}^2$.



Note the orientation of the axes. If you sat on top of z-axis and looked down, you would see the usual orientation for the x and y axes.

Graphs of functions of two variables

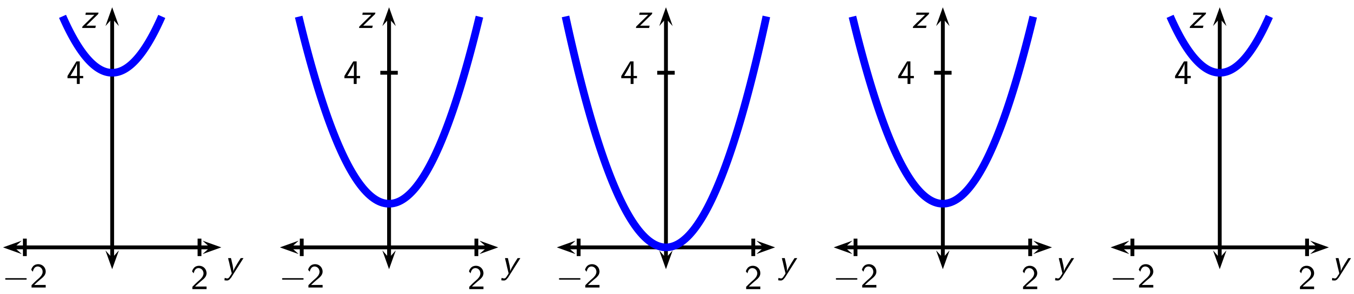
Given a function of two variables, how can we visualise it?

For example, what does the graph of

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x, y) = x^2 + y^2$$

look like. That is, we want to sketch the set of points in $\mathbb{R}^3$ satisfying $z = f(x, y)$.

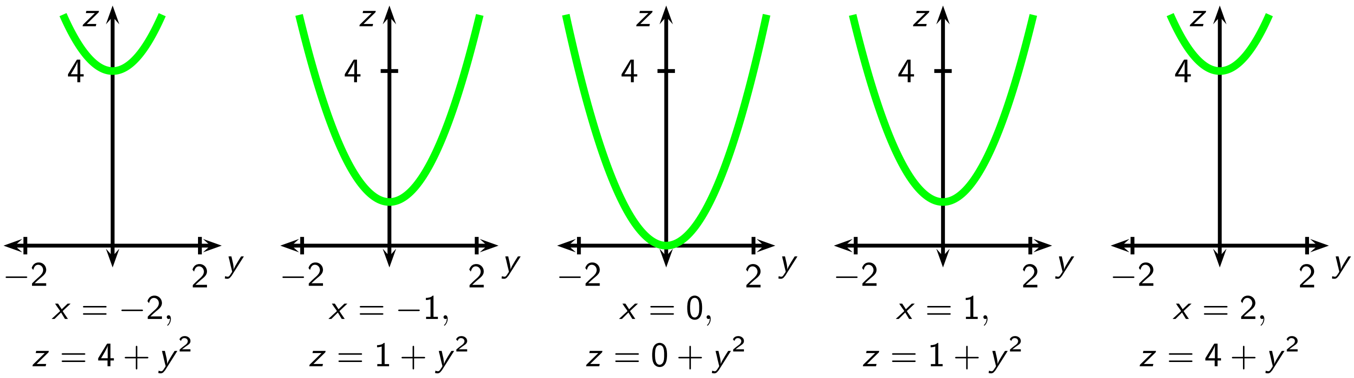
Let's start by looking at some vertical slices with constant x.

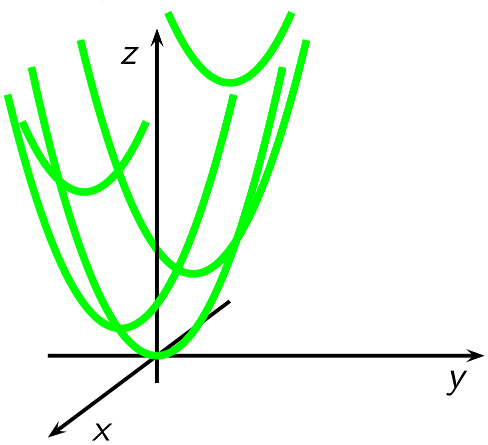


$x = -2$, $z = 4 + y^2$ | $x = -1$, $z = 1 + y^2$ | $x = 0$, $z = 0 + y^2$ | $x = 1$, $z = 1 + y^2$ | $x = 2$, $z = 4 + y^2$

Next put these together.

Graphs of functions of two variables



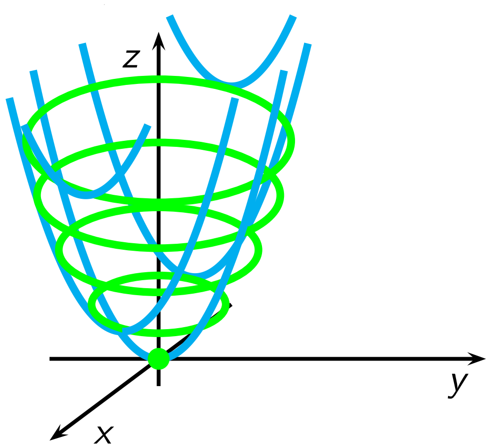


We could also take slices of constant y.
Try plotting these yourself.

Horizontal slices

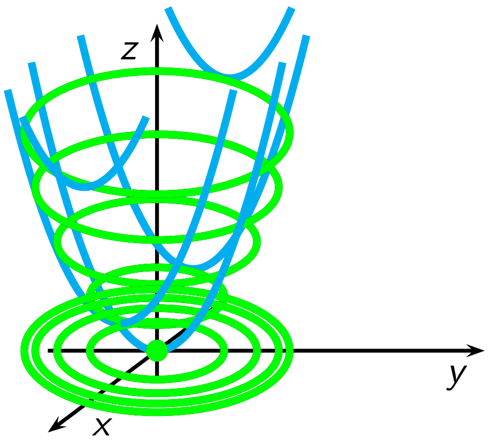
We could also take horizontal slices, that is, slices of constant z.

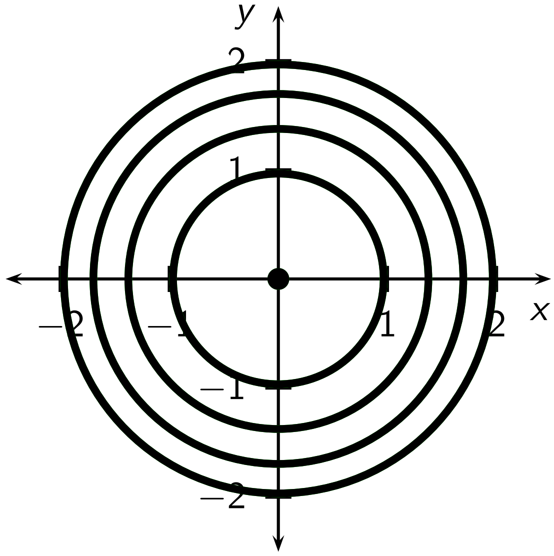
$$
\begin{array}{lllll}
z & = & -1 & : & \text{no solution} \\
z & = & 0 & : & (x,y) = (0,0) \quad \text{a single point} \\
z & = & 1 & : & x^2 + y^2 = 1 \quad \text{a circle of radius } 1 \\
z & = & 2 & : & x^2 + y^2 = 2 \quad \text{a circle of radius } \sqrt{2} \\
 & \text{etc} & & &
\end{array}
$$



Horizontal slices

If we plot the horizontal slices in the xy-plane, we have a contour map.





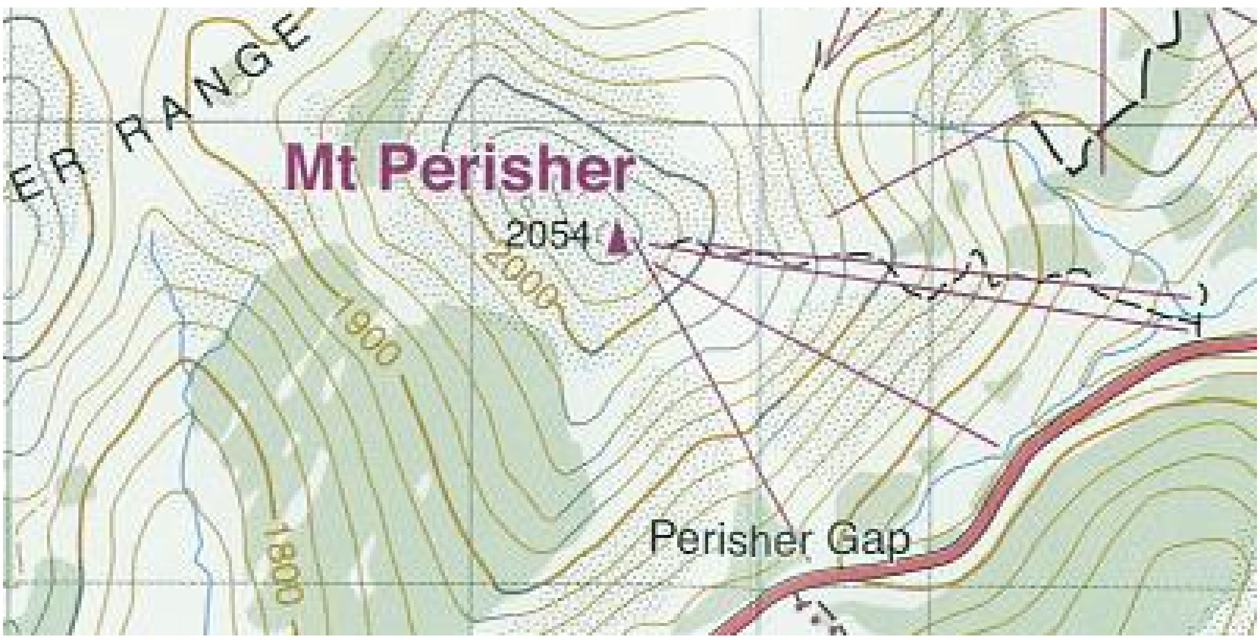
We have plotted some level curves or contours of f.

Contours or other slices are a good way of visualising a surface.

Level curves - examples

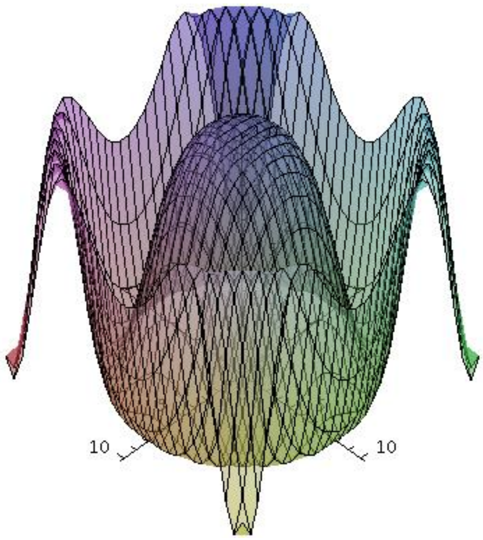
Contours on topographical maps are used to describe a surface. Maps Downunder have some sample maps on their website.

`http://www.mapsdownunder.com.au/cgi-bin/mapshop/ABC-MTPKT.html`

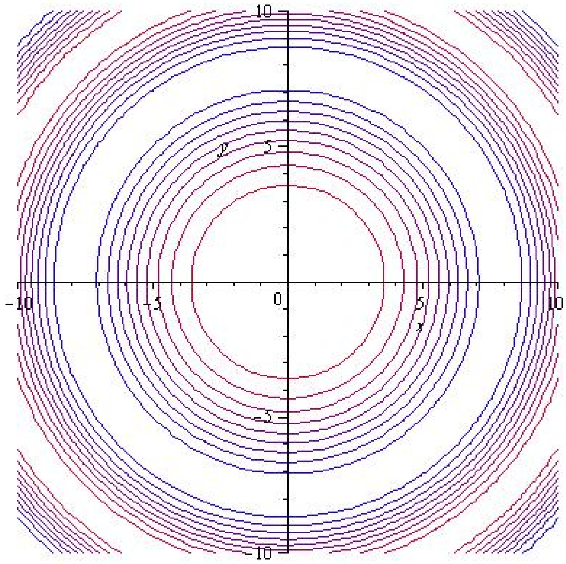


Level curves - examples

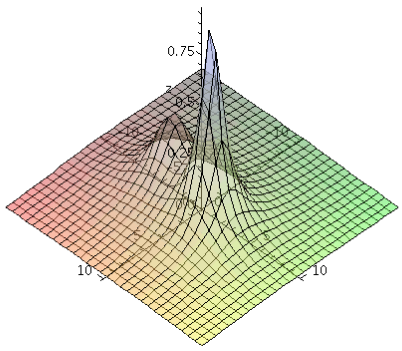
```
> f1 := 1-sin((x^2+y^2)/40)^2:
> with(plots):
  plot3d(f1,x=-10..10,y=-10..10,
  axes=normal,transparency=0.5,
  labels=[x,y,z],grid=[30,30]);
```

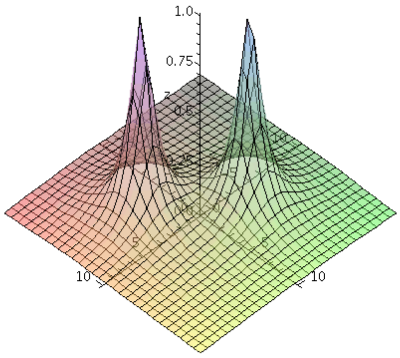


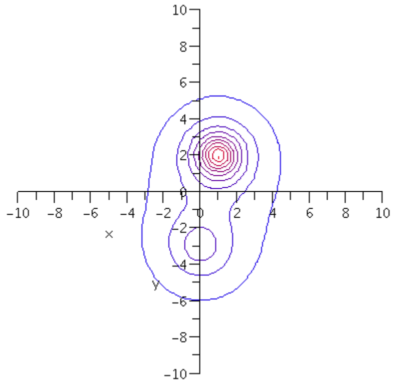

```
> contourplot(f1,x=-10..10,y=-10..10,
  grid=[50,50],view=[-10..10,-10..10],
  contours=[0,0.1,0.2,0.3,0.4,0.5,0.6,
  0.7,0.8,0.9,1],coloring=[blue,red]);
```

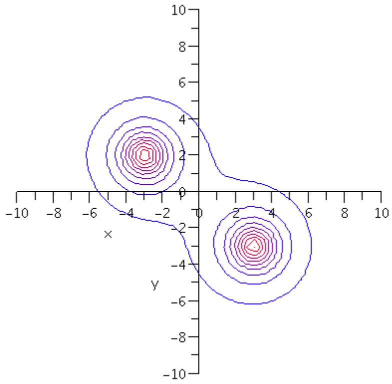


Level curves - examples

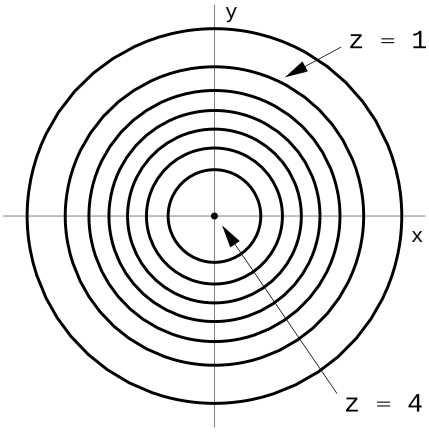






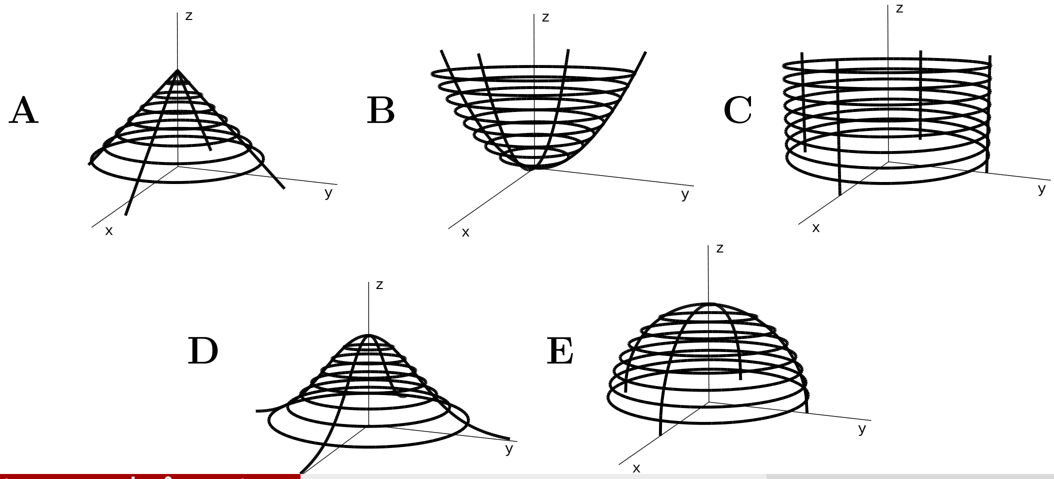


Level curves - example



Let f be a function of two variables. The $f(x,y) = 0.5,\ 1.0,\ 1.5,\ 2.0,\ 2.5,\ 3.0,\ 3.5,\ 4.0$ level curves are draw on the left.

Which of the surfaces below could be the graph $z = f(x,y)$? Give reasons for your choice



Surfaces - an example

Sketch the level curves of

$$f(x,y) = 4 - \sqrt{x^2 + y^2}$$

and describe the surface $z = f(x,y)$.

$z = f(x,y) = 4 - \sqrt{x^2+y^2}$

level curves are $c = 4 - \sqrt{x^2+y^2}$

$\Rightarrow\ c - 4 = -\sqrt{x^2+y^2}$ (Note $c-4 \le 0 \Rightarrow c \le 4$)

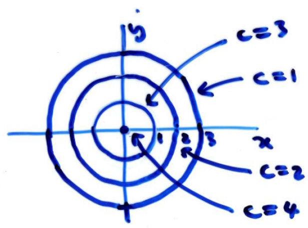
$c=4$: $(x,y) = (0,0)$

$c=3$: $x^2+y^2 = 1$ (circle of radius 1)

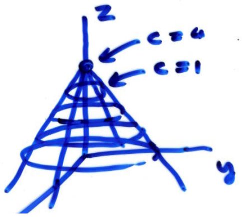
$c=2$: $x^2+y^2 = 4$ (" " " 2)

$c=1$: $x^2+y^2 = 9$ (" " " 3)

:



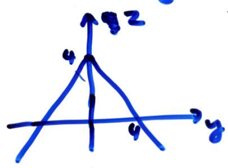
This is a cone with vertex at $(0,0,4)$



$x=0$ slice:

$z = 4 - \sqrt{0+y^2}$

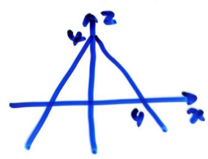
$= 4 - |y|$



$y=0$ slice:

$z = 4 - \sqrt{x^2+0}$

$= 4 - |x|$



Surfaces - an example

Sketch the level curves of

$$f(x, y) = \sqrt{1 - x^2 - 3y^2}$$

and describe the surface $z = f(x, y)$.

$$z = f(x,y) = \sqrt{1-x^2-3y^2}$$

level curves are $c = \sqrt{1-x^2-3y^2}$

$$\Rightarrow \quad x^2+3y^2 = 1-c^2 \qquad (c \geq 0)$$

$$\left.\begin{array}{ll} c=1: & (x,y)=(0,0) \\ c=\frac{3}{4}: & x^2+3y^2 = \frac{7}{16} \\ c=\frac{1}{2}: & x^2+3y^2 = \frac{3}{4} \\ c=\frac{1}{4}: & x^2+3y^2 = \frac{15}{16} \\ c=0: & x^2+3y^2 = 1 \end{array}\right\} \text{ellipses}$$